Optimal Resonance Forcing of Nonlinear Systems

[a,b]Glenn Foster, [a,b]Alfred Hübler, and [a]Karin Dahmen

University of Illinois at Urbana-Champaign
[a]*Department of Physics and The Materials Computation Center*
[b]*Center for Complex Systems Research*

gfoster@uiuc.edu

"*Resonant forcing of multi-dimensional chaotic map systems*" GF,AH,KD
submitted to Physical Review E

"*Resonance spectroscopy with chaotic forcing functions*" GF,AH,KD
in preparation

Sympatico work:
"*Enhanced Diffraction Pattern from a Fibonacci Chain*" Jian Xu & AH
PRB, **67**, 184202 (2003)

Overview

- We study the response to aperiodic forcings of nonlinear map dynamics and derive the **optimal** forcing dynamics using a discrete variation.

- We find that the optimal forcing is a transformation of the natural dynamics of the system

- We find that for 1-D linear and nearly-linear systems the dynamics of the optimal forcing can be fairly characterized as the **time-reversed** unperturbed dynamics of the system.

- We apply optimal forcings for system identification: resonance spectroscopy on nonlinear maps

- Under these metrics and small perturbations we find a conserved quantity; force • resulting displacement = constant for each time step.

- Computational code to go into Materials Computation Center software archive

Introduction

Of general interest is the response, resonance, and robustness of general dynamical systems to a forcing input.

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Questions:

- What input to these systems will cause the largest response?
- What input should these systems be protected from?
- Or from a robust engineering perspective:
  What is the worst possible thing to do to this system?

Practical limits on what forcing inputs are accessible so focus on getting the **largest** response using a **minimum** amount of effort.

We study

- **optimal** forcings of **nonlinear** (including chaotic) **map dynamics**

Focus on forcing that is *optimal* in the sense that it gives the largest response with the smallest measure of forcing input (under some metric).

The general system of interest

Dynamical system: current state determined by at most m previous states.

Evolution of system is

- deterministic
- noise free
- autonomous

i.e., deterministic dynamical system in discrete time (delay difference eq.)

(**natural dynamics**) $y_n = \mathrm{f}(y_{n-1}, y_{n-2}, \ldots y_{n-m})$

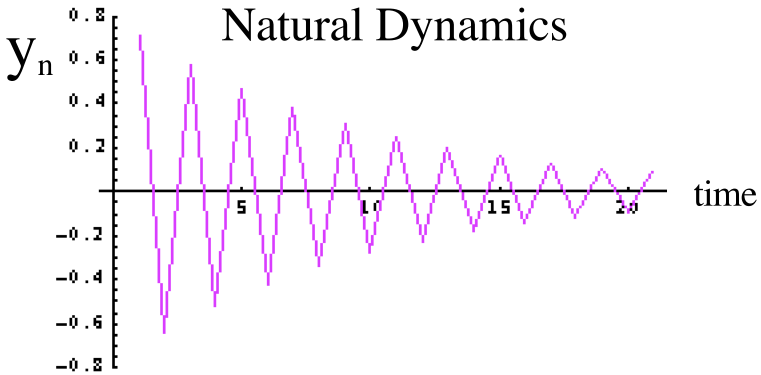
We wish to compare the evolution of this system with one we additively force:

(**forced dynamics**) $x_n = \mathrm{f}(x_{n-1}, x_{n-2}, \ldots, x_{n-m}) + F_{n-1}$

(initial conditions are identical, $x_n = y_n$ for $n = -m, .., 0$)

Visually:

$$y_{n+1} = \mathrm{f}(y_n, y_{n-1}, \ldots)$$



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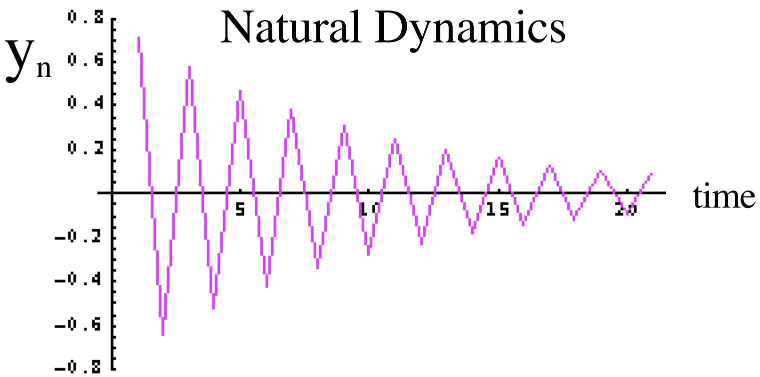

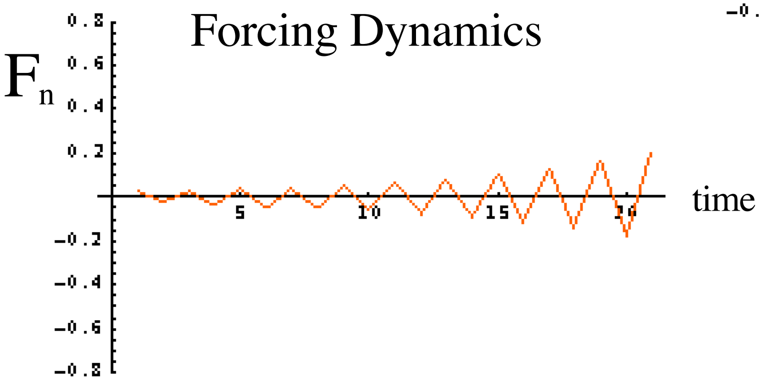

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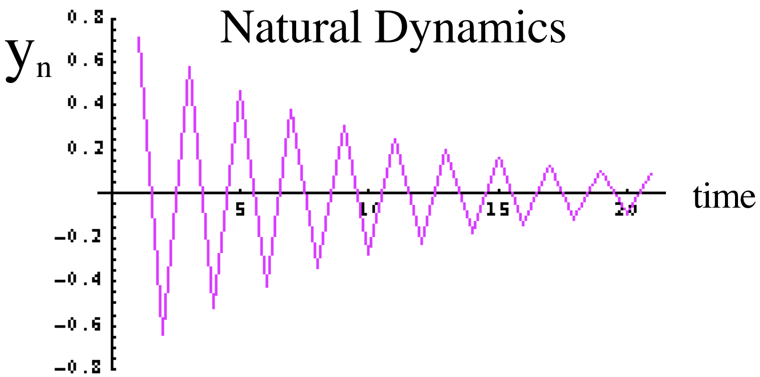
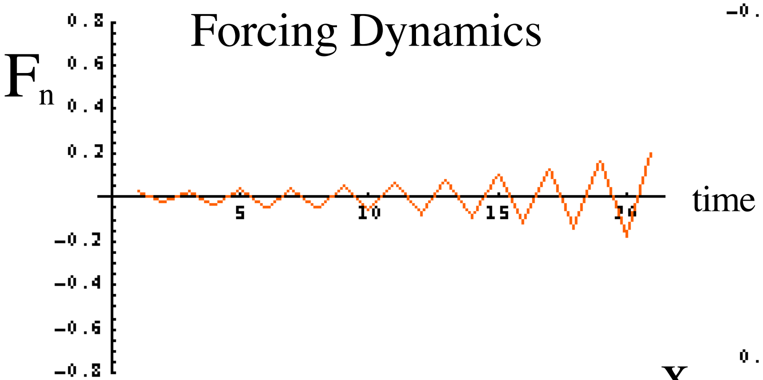
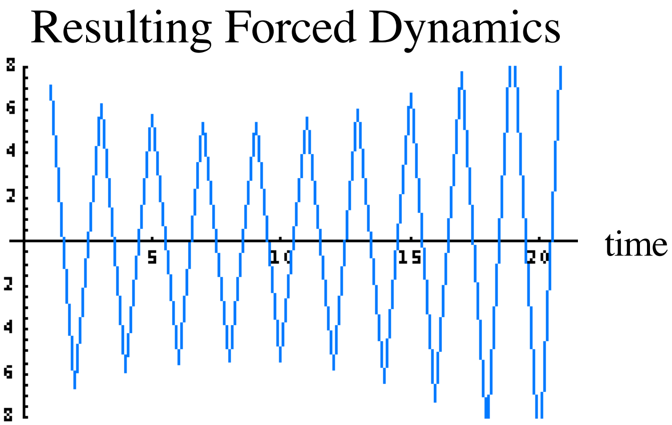

Natural Dynamics


Forcing Dynamics


Resulting Forced Dynamics

$$x_{n+1} = \mathrm{f}(x_n, x_{n-1}, \ldots) + F_n$$

Metrics (minimum effort, maximum response)

We force the system for N timesteps $F_0 \ldots F_{N-1}$ and look at the response after the last forcing.

We wish to constrain the **mean square average forcing**

$$\left\langle F^2 \right\rangle = \frac{1}{N} \sum_{n=0}^{N-1} F_n^2$$

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How to calculate optimal forcing subject to constraint?

We define a functional

$$L = R^2 - \xi\left(\langle F^2 \rangle - c\right) - \sum_{n=0}^{N-1} \mu_n\left(x_{n+1} - f\left(x_n, \ldots, x_{n-m}\right) - F_n\right)$$

1-D system with time delay

Take variation, eliminate Lagrange multipliers

We take the variation; solve the resulting equations to eliminate Lagrange multipliers, and thereby derive a formula for the **optimal forcing dynamics**:

$$F_{n-1} = \sum_{j=0}^{m} F_{n+j} \frac{\partial f}{\partial x_{n-j}}$$

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This formula greatly simplifies for systems like constant coefficient delay equations or to illustrate today, a 1-D system without delay; $y_{n+1} = \mathrm{f}(y_n)$

$$F_{n-1} = F_n f'(x_n)$$

Resonance spectroscopy and system identification

- experimental system is forced (**resonant** F_n)

$$x_{n+1} = \mathrm{f}_\mathrm{a}(x_n) + F_n(\mathrm{b})$$

$$F_n = F_{n-1} / f_b'(z_\mathrm{n})$$

- resulting output measured

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Compare output to model

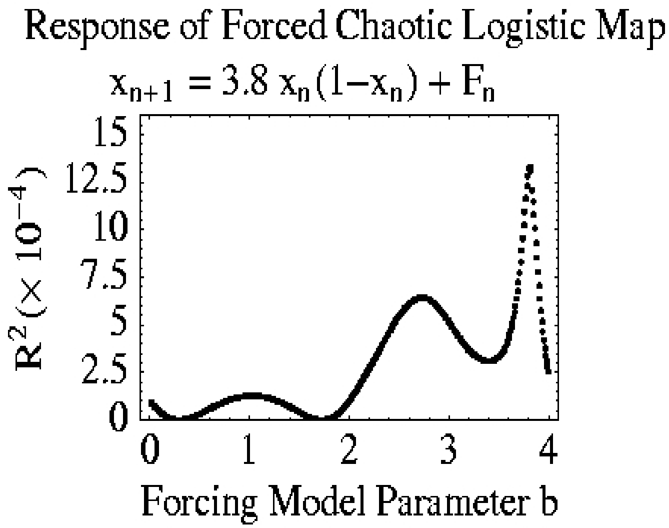
$$_{b}x_{n+1} = f_b({}_{b}x_n) + F_n(b)$$

When model and output match and **if** resonant forcing for experimental system was among those applied, when we measure response $R_b^2 = (x_N - y_N)^2$,we find that:

- can be no higher response than that of matching forcing
- guarantee of good signal to noise

System identification of the chaotic logistic map

- **logistic map** $y_{n+1} = a\, y_n(1-y_n)$ chaotic for certain values of a (e.g. 3.8)
- nonlinear f, so $F_n = \dfrac{F_{n-1}}{f'(x_n)}$
- $\{x_n\}$, $f'(x_n)$ are chaotic



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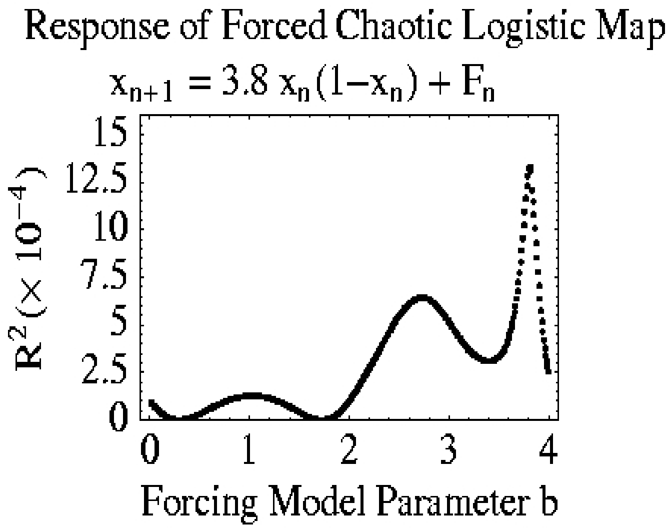

Assume:

- experimental system has unknown *a*
- have a reasonable model: $z_{n+1} = b\, z_n(1-z_n)$
- have access to system (can set initial condition or patience/luck)

Prior to forcing:

- Calculate optimal forcing for each *b* value using model:
- $F_{n+1} = F_n / f'(z_{n+1}) = F_n / b(1-2z_{n+1})$
- Iterate forcing numerically, then rescale to satisfy $\langle F^2 \rangle$

Stability of control and a conserved quantity:

Our optimal forcing can be viewed as a control:

- stable if the displacement between nearby trajectories decreases on average.
- chaotic case, control will be unstable (but will work for a short number time steps)

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For small displacements, the distance **d** between two neighboring trajectories will evolve as

$$\mathbf{d}_{n+1} = \mathbf{J}_n \cdot \mathbf{d}_n$$

(and the Jacobian will be locally constant) but since we know the dynamics of optimal forces evolve as

$$\mathbf{F}_{n-1} = \mathbf{J}(\mathbf{x}_n)^T \cdot \mathbf{F}_n$$

then the product of the force and displacement at each timestep is

$$\mathbf{F}_{n-1} \cdot \mathbf{d}_{n-1} = \mathbf{F}_n \cdot \mathbf{d}_n = \text{constant}$$

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Summary

- Found that for 1-D linear and nearly-linear systems the dynamics of the optimal forcing is related to the unperturbed dynamics of the system. (negative of the Lyapunov exponent, frequency matched)

- Applied optimal forcings for system identification: resonance spectroscopy on nonlinear maps. Even works in the short term on chaotic systems.

- In particular, derived optimal forcing dynamics for
  - multi-dimensional systems
  - multi-delay systems

  under the constraints of minimizing the average square force and maximizing the response.

- Under these metrics and small perturbations we find a conserved quantity; the force multiplied by the resulting displacement is constant for each time step.