Introduction to Multigrid

Waves on different grids

Consider wave with wavenumber k:

$$u_{j,k}^h = \sin\frac{jk\pi}{n} \text{ on grid } \Omega^h$$

Consider same wave on the coarse grid (half the points), by simply taking value at every other point:

$$u_{j,k}^{2h} = u_{2j,k}^h = \sin\frac{2jk\pi}{n} = \sin\frac{jk\pi}{n/2} \text{ on grid } \Omega^{2h}.$$

So kth mode on fine grid gives kth mode on coarse grid.

Should not be surprising since kth mode is wave with wavelenght $\frac{2}{k}$, which does not depend on n.
Another way to see this is that we have half the number of points and half the number of waves.

Waves on different grids

The oscillatory waves on Ω^{2h}, which has $n/2$ points, are waves with wavenumbers $n/4 \leq k < n/2 - 1$.

Since wave number does not change, the modes with wavenumber $n/4 \leq k < n/2$ become oscillatory. So half the smooth modes on Ω^h become oscillatory on Ω^{2h}. The other half remain smooth (but less smooth than before).

The oscillatory modes on the fine grid cannot be represented on the coarse grid.

What happens to the oscillatory modes on Ω^h when we go to Ω^{2h}?

Why is this a problem for our solution algorithm?

Waves on different grids

Given problem with $n = 32$ with gives **33 grid points** and $j = 0 \ldots 32$.

We look at following waves on $\Omega^{1/32}$ and $\Omega^{1/16}$:
1) $k = 5, 14, 26$
2) $k = 5, 14, 30$

Notice that
$k = 5$ is smooth on both grids (but less smooth on $\Omega^{1/16}$)
$k = 14$ is smooth on $\Omega^{1/32}$ and becomes oscillatory on $\Omega^{1/16}$
$k = 26, 30$ are oscillatory on $\Omega^{1/32}$ and become smooth on $\Omega^{1/16}$

The effect for $k = 26, 30$ is called aliasing: an oscillatory wave disguises itself as a smooth one. In fact a wave with wavenumber $k > n - 1$ appears as a wave with wavenumber $k = 2n - k$.

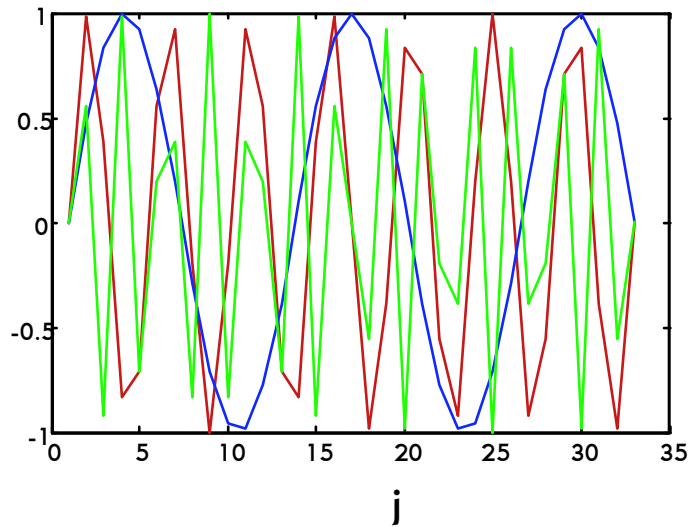There is no way to distinguish the two on the given grid.

Waves on different grids

$$\sin \frac{jk\pi}{n}$$

k1=5
k2=14
k3=26
n=32

Waves on different grids

$$\sin \frac{jk\pi}{n/2}$$

k1=5
k2=14
k3=26
n=16
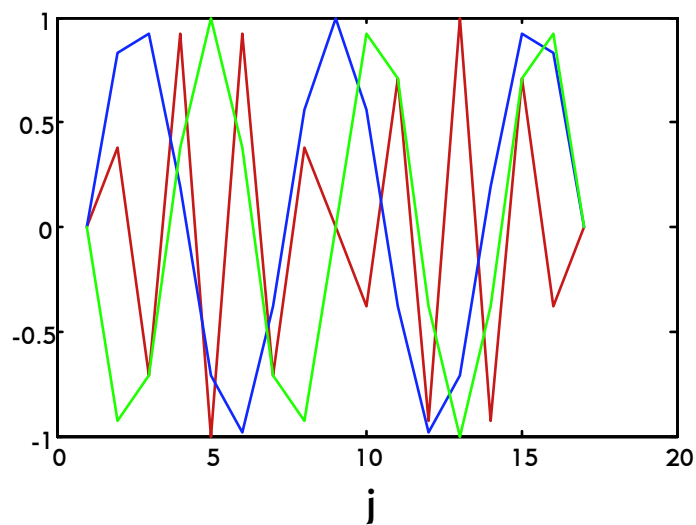
Waves on different grids

$$\sin \frac{jk\pi}{n}$$

k1=5
k2=14
k3=30
n=32
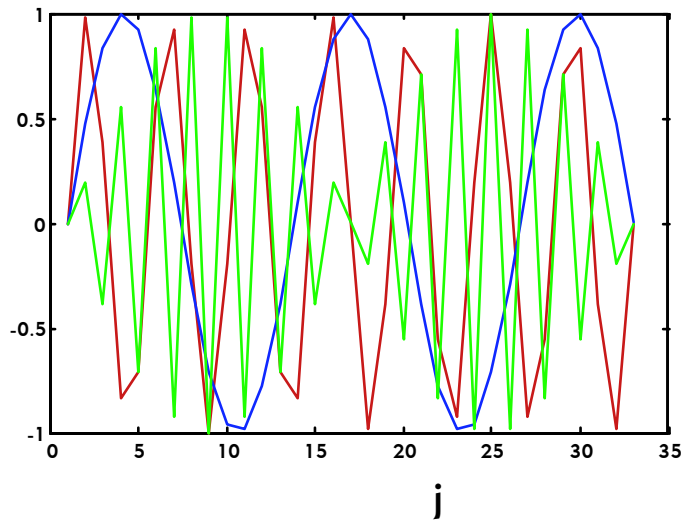
Waves on different grids

$$\sin \frac{jk\pi}{n/2}$$

k1=5
k2=14
k3=30
n=16
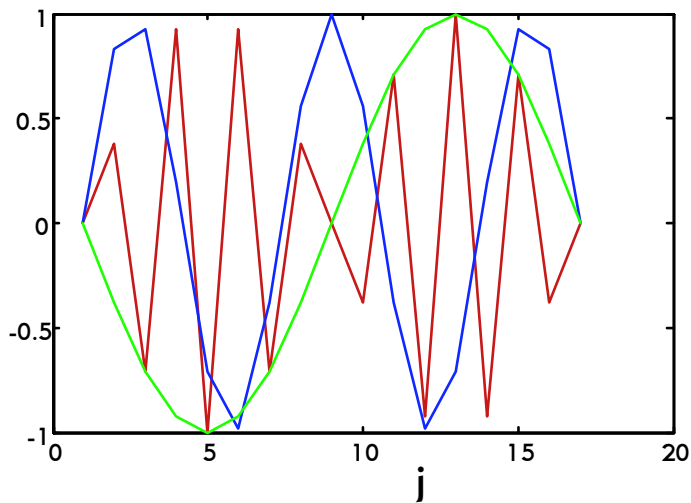
The basic MG idea

We saw that our basic iteration or relaxation methods are good in damping the oscillatory components of the error. They are not effective on the smooth modes.

So the idea is to use relaxation on the fine grid until oscillatory modes are sufficiently reduced and convergence slows down.

Then we move to the coarse grid where the (fine grid) smooth modes are more oscillatory and hence more effectively reduced by relaxation.

We saw that the very smooth modes ($k < n/4$) on the fine grid remain smooth on the coarse grid. Hence they will still be reduced slowly by relaxation.

The natural answer to this is to move to the next coarser grid, etc.

Typically at some point a direct solver is used.

Grid transfer and coarse grid eq.s

Important question now is how do we move among the grids.

1) How do we use the solution from the coarser grids (that have reduced or removed the smooth components of error) to get a better solution on the fine grid?

2) How do we transfer the approximate solution from fine grid to coarse grid and how do we derive the equations to solve on the coarse grid?

Grid transfer and coarse grid eq.s

Let's derive equations for the coarse grid.

We want to compute a correction to the solution on the fine grid. That means we must compute an approximation to the error.

The residual gives us an equation to compute the error: $Ae = r$

So we compute the residual at Ω^h and map it to Ω^{2h}. This can be done in many ways, but for the moment we stick to the simple trivial injection we saw before: $r_j^{2h} = r_{2j}^h$. This will provide our right hand side.

Again to keep things simple we assume that the coarse grid operator is the coarse grid discretization of the PDE (basically same operator as on the fine grid): A^{2h} (fine grid A^h).

So we have the equation: $A^{2h}e^{2h} = r^{2h}$

Grid transfer and coarse grid eq.s

Suppose we find a satisfactory approximation to the error e^{2h}. How do we use this to correct the fine grid approximation to the solution?

This operation is called interpolation or prolongation. Again many possibilities exist. We will look at a simple one first. Clearly for the points existing in both grids we can simply take the same values (just as going from fine to coarse grid). For the intermediate points we use linear interpolation. An important rationale for this is the following.

Assume the error on the fine grid is smooth. Further assume that we have the exact coarse grid error on the coarse grid. Linear interpolation of the coarse grid error on the fine grid will give a smooth function. Hence we may assume we get a good approximation to the fine grid error.

On the other hand if the fine grid error is oscillatory we cannot hope to get a good approximation (hence e^h should be smooth).

Grid transfer and coarse grid eq.s

The interpolation or prolongation operator is represented by I_{2h}^h.
We will represent the restriction operator by I_h^{2h}.

Assume n is even, $(n+1)$ grid points, and we ignore the boundary points and assume zero (Dirichlet) boundary conditions. Then we need to map from coarse grid points $1...\frac{n}{2}-1$ to fine grid points $1...n-1$:

$$I_{2h}^h : \mathbb{R}^{\frac{n}{2}-1} \rightarrow \mathbb{R}^{n-1} = \frac{1}{2} \begin{pmatrix} 1 & & \\ 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & \\ & & \ddots \end{pmatrix}$$

The operator takes the coarse grid value at even grid points and averages values at neighboring points for the odd points. If we include the points on the boundary it would copy the values (even points).

Grid transfer and coarse grid eq.s

We will discuss two restriction operators: I_h^{2h}
Injection simply assigns the fine grid values to the corresponding coarse grid variables:

$$I_h^{2h} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{\frac{n}{2}-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}$$

Another restriction operator is so-called full-weighting.
Here the values at the coarse grid points are taken as weighted averages of their (immediate) neighbors. The full weighting operator is a scaled transpose of the linear interpolation (prolongation) operator.

(compare with the domain decomposition operators)

Grid transfer and coarse grid eq.s

Full weighting at gives for a coarse grid variable:

$$v_j^{2h} = \tfrac{1}{4}\left(v_{2j-1}^h + 2v_{2j}^h + v_{2j+1}^h\right)$$

Written as a matrix operator (for whole grid) this gives

$$I_h^{2h} : \mathbb{R}^{n-1} \to \mathbb{R}^{\frac{n}{2}-1} = \tfrac{1}{4}\begin{pmatrix} 1 & 2 & 1 & & & \\ & & 1 & 2 & 1 & \\ & & & & & \ddots \end{pmatrix}$$

Note that all these steps are very easy to implement as local operators at each point.

2-Grid algorithm

We now have all the steps to put our first multigrid algorithms together.

Two grid algorithm: $Au = f$ (on Ω^h)
Ω^h: Do k_1^h relaxations on $A^h u^h = f^h$ starting with some initial guess
$\quad r^h = f^h - A^h u^h; \quad r^{2h} = I_h^{2h} r^h;$

Ω^{2h}: Solve (directly) $A^{2h} e^{2h} = r^{2h}$
$\quad e^h = I_{2h}^h e^{2h}; \quad u^h = u^h + e^h;$

Ω^h: Do k_2^h relaxations on $A^h u^h = f^h$ starting with new u^h.

A direct solver is used for $A^{2h} e^{2h} = r^{2h}$. Obviously for very large systems this is not practical. However, the two grid algorithm is often used for analysis.

An obvious alternative for a direct solver is to do another coarse grid correction on Ω^{4h} and so on (CS students feel the recursion coming).

2-Grid algorithm

We may also use an iterative solver (another couple of relaxation sweeps) on $\Omega^{2h}: A^{2h}e^{2h} = r^{2h}$. This can in fact already be surprisingly effective.

We typically use a direct solver at some level (when the number of variables gets sufficiently small).

V-cycle variations

V-cycle MG algorithm: $Au = f$ (on Ω^h)

Ω^h: Do k_1^h relaxations on $A^h u^h = f^h$ starting with some initial guess
$\quad r^h = f^h - A^h u^h; \quad f^{2h} = I_h^{2h} r^h;$

Ω^{2h}: Do k_1^{2h} relaxations on $A^{2h} u^{2h} = f^{2h}$ starting with zero guess
$\quad r^{2h} = f^{2h} - A^{2h} u^{2h}; \quad f^{4h} = I_{2h}^{4h} r^{2h};$

and so on for coarser and coarser grids

Ω^{Lh}: Solve (directly) $A^{Lh} u^{Lh} = f^{Lh};$ (suff. iterations make direct solve)
$\quad e^{(L-1)h} = I_{Lh}^{(L-1)h} u^{Lh};$

$\Omega^{(L-1)h}: u^{(L-1)h} = u^{(L-1)h} + e^{(L-1)h};$
$\quad$ Do $k_2^{(L-1)h}$ relaxations on $A^{(L-1)h} u^{(L-1)h} = f^{(L-1)h}$ using new $u^{(L-1)h}$

and so on for finer and finer grids

$\Omega^h: u^h = u^h + e^h;$
$\quad$ Do k_2^h relaxations on $A^h u^h = f^h$ starting with new u^h.

V-cycle variations

Let's experiment:

Local mode analysis

How to analyze smoothing behavior for more general problems and relaxations: local mode (normal mode/Fourier) analysis.
In general computing eigenvectors/values is too hard (harder than solvng a linear system)

The idea is to derive the smoothing factor from idealization of the equations at a point (experiment to find worst point).

This separates the smoothing of the error from the other algorithmic components. It also provides a optimal figure against which to compare overall performance of algorithm.

Idealizations:
• Assume infinite domain
• Assume equations are same everywhere
• Assume relaxation scheme is linear process

Local mode analysis

Error given by linear iteration: $e^{(k+1)} = Ge^{(k)}$

Now we assume the error consists of Fourier modes and we analyze how relaxation acts on these modes (ideally no mixing of modes). **We should analyze the damping of the eigenvectors of the iteration matrix.** However, we assume that oscillating modes are approximately eigenvectors (generally true); and apply smoothing analysis to these.

For discrete domain we have waves $w_j = \sin\frac{jk\pi}{n}$ with wavenumber $k = 1\ldots n$.
Now we replace $\frac{k\pi}{n}$ by θ (continuous wavenumber) and consider waves $w_j = \exp(ij\theta)$ with $\theta \in (-\pi, \pi]$.

Values $|\theta|$ near 0 correspond to low frequency waves; values of $|\theta|$ near π correspond to high frequency waves.

Wavelength of mode θ is $\frac{2\pi h}{|\theta|}$.

Local mode analysis

Error at point j at iteration k : $e_j^{(k)} = A(k)\exp(ij\theta),\ -\pi < \theta \le \pi$.

Goal is to find relation $A(k+1) = A(k)G(\theta)$,
where $G(\theta)$ is called the amplification factor (for mode θ)

This way we can analyze the convergence of the modes separately.
For convergence (of relaxation method) we need $|G(\theta)| < 1$ for all θ.

For MG only need damping of oscillatory modes: $|G(\theta)| < 1,\ \frac{\pi}{2} \le |\theta| \le \pi$.

We define smoothing factor as $\mu = \max_{\frac{\pi}{2}\le|\theta|\le\pi}|G(\pi)|$.
(slowest damping of oscillatory modes)

1D Example: Jacobi

Consider one-dimensional equation $-u_{xx} + cu = f$

Central finite differences gives: $-v_{j-1} + (2 + h^2 c_j)v_j - v_{j+1} = h^2 f_j$

Jacobi: $v_j^{(k+1)} = \frac{1}{(2+h^2 c_j)}\left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)}\right)$

weighted Jacoby: $v_j^{(k+1)} = \frac{\omega}{(2+h^2 c_j)}\left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)}\right) + (1-\omega)v_j^{(k)}$

For the error we get:

$$e_j^{(k+1)} = u_j - v_j^{(k+1)} = \omega u_j - \frac{\omega}{(2+h^2 c_j)}\left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)}\right) + (1-\omega)u_j - (1-\omega)v_j^{(k)}$$

Since u satisfies the equation we have $u_j = \frac{1}{(2+h^2 c_j)}(h^2 f_j + u_{j-1} + u_{j+1})$

1D Example: Jacobi

Substituting $u_j = \frac{1}{(2+h^2 c_j)}(h^2 f_j + u_{j-1} + u_{j+1})$ in expression for error

$$e_j^{(k+1)} = \omega u_j - \frac{\omega}{(2+h^2 c_j)}\left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)}\right) + (1-\omega)u_j - (1-\omega)v_j^{(k)}$$

gives

$$e_j^{(k+1)} = \frac{\omega}{(2+h^2 c_j)}(h^2 f_j + u_{j-1} + u_{j+1}) - \frac{\omega}{(2+h^2 c_j)}\left(h^2 f_j + v_{j-1}^{(k)} + v_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$$

and finally

$$e_j^{(k+1)} = \frac{\omega}{(2+h^2 c_j)}\left(e_{j-1}^{(k)} + e_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$$

1D Example: Jacobi

Now substitute $A(k)\exp(ij\theta)$ for $e_j^{(k)}$ and simplify expression
First we assume $c = 0$.

$e_j^{(k+1)} = \frac{\omega}{2}\left(e_{j-1}^{(k)} + e_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$ becomes

$A(k+1)\exp(ij\theta) =$

$\qquad \frac{\omega}{2}(A(k)\exp(\iota(j-1)\theta) + A(k)\exp(\iota(j+1)\theta)) + (1-\omega)A(k)\exp(ij\theta)$

$A(k)\exp(\iota[j-1]\theta) + A(k)\exp(\iota[j+1]\theta) = A(k)\exp(ij\theta)[\exp(-\iota\theta) + \exp(\iota\theta)] =$

$\qquad A(k)\exp(ij\theta)[\cos\theta - \iota\sin\theta + \cos\theta + \iota\sin\theta] =$

$\qquad A(k)2\exp(ij\theta)\cos\theta$

$A(k+1)\exp(ij\theta) = \frac{\omega}{2}A(k)2\exp(ij\theta)\cos\theta + (1-\omega)A(k)\exp(ij\theta) =$

$\qquad\qquad A(k)\exp(ij\theta)[1 - \omega(1-\cos\theta)]$

1D Example: Jacobi

Using $\cos\theta = \cos 2 \cdot \left(\frac{\theta}{2}\right) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$ we get

$A(k)\exp(ij\theta)[1 - \omega(1-\cos\theta)] = A(k)\exp(ij\theta)[1 - 2\omega\sin^2\left(\frac{\theta}{2}\right)]$

This gives the amplification factor

$A(k+1) = A(k)[1 - 2\omega\sin^2\left(\frac{\theta}{2}\right)] = A(k)G(\theta)$, for $-\pi < \theta \leq \pi$.

These are the same convergence rates we saw for discrete wavenumbers k, corresponding to $\theta_k = \frac{k\pi}{n}$.
This is not generally the case, however.
From the above result we know that optimal weight is $\omega = \frac{2}{3}$, which yields

$\qquad \mu = G\left(\frac{\pi}{2}\right) = |G(\pm\theta)| = \frac{1}{3}$

1D Example: Jacobi

Now we assume $c \neq 0$.

$e_j^{(k+1)} = \frac{\omega}{2+h^2 c_j}\left(e_{j-1}^{(k)} + e_{j+1}^{(k)}\right) + (1-\omega)e_j^{(k)}$ **becomes**

$$A(k+1)\exp(\imath j\theta) = \frac{\omega}{2+h^2 c_j}A(k)2\exp(\imath j\theta)\cos\theta + (1-\omega)A(k)\exp(\imath j\theta) =$$
$$A(k)\exp(\imath j\theta)\left[1-\omega+\frac{2\omega\cos\theta}{2+h^2 c_j}\right] =$$
$$A(k)\exp(\imath j\theta)\left[1-\omega\left(1-\frac{2\cos\theta}{2+h^2 c_j}\right)\right]$$

Result depends on c_j. Typically take some fixed c for analysis: maximum or minimum c_j or worst case for amplification factor.

This gives for the amplification factor: $G(\theta) = 1-\omega\left(1-\frac{2\cos\theta}{2+h^2 c}\right)$ **this can be rewritten (for comparison) as**

$$G(\theta) = 1-\omega(1-\cos\theta)+\omega(1-\cos\theta)-\omega\left(1-\frac{2\cos\theta}{2+h^2 c}\right)$$

1D Example: Jacobi

The amplification factor $G(\theta) = 1-\omega\left(1-\frac{2\cos\theta}{2+h^2 c}\right)$ **can be rewritten (for comparison) as**

$$G(\theta) = 1-\omega(1-\cos\theta)+\omega(1-\cos\theta)-\omega\left(1-\frac{2\cos\theta}{2+h^2 c}\right) =$$
$$G_0(\theta)+\omega\left(1-\cos\theta-1+\frac{2\cos\theta}{2+h^2 c}\right) =$$
$$G_0(\theta)+\omega\left(-1+\frac{2}{2+h^2 c}\right)\cos\theta =$$
$$G_0(\theta)-\frac{\omega h^2 c}{2+h^2 c}\cos\theta$$

where $G_0(\theta)$ is $G(\theta)$ for the case $c = 0$.

$G(\theta)$ **only differs significantly from** $G_0(\theta)$ **if $h^2 c$ is not too small.**

Note result is different from book p.52

1D Example: Gauss-Seidel

For Gauss-Seidel we get

$$e_j^{(k+1)} = \frac{e_{j-1}^{(k+1)} + e_{j+1}^{(k)}}{2 + h^2 c_j},$$

assuming carry out the relaxations from left to right (increasing j).

Again assuming $e_j^{(k)} = A(k)\exp(\imath j\theta)$ **and** $c = 0$ **we get**

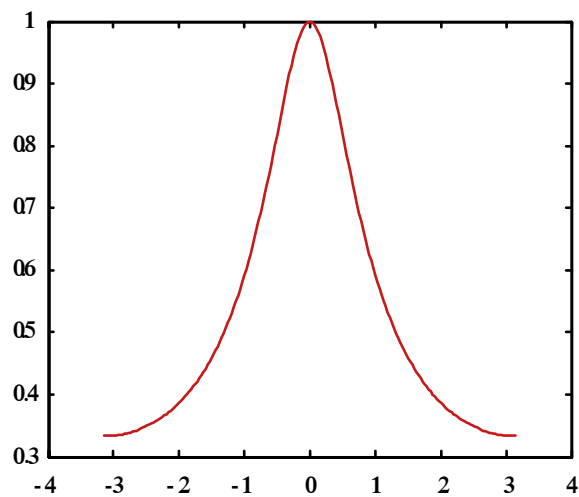$$A(k+1) = \frac{\exp(\imath\theta)}{2 - \exp(-\imath\theta)} A(k).$$

So we have as amplification factor $G(\theta) = \frac{\exp(\imath\theta)}{2 - \exp(-\imath\theta)}$

The graph of $|G(\theta)|$ **shows the smoothing factor is obtained for** $\theta = \frac{\pi}{2}$.

$$\mu = |G(\tfrac{\pi}{2})| = \left|\tfrac{i}{2+i}\right| = \sqrt{\tfrac{1}{5}} \approx 0.453$$

1D Example: Gauss-Seidel

1D Example: Gauss-Seidel

Contrary to what we saw for the Jacobi iteration, we notice that in this case the eigenvalues on the infinite domain without boundaries are different from the case with boundaries.

So the result only approximately gives the smoothing factors on finite domains. As we can verify from the results from chapter 2.

Local mode analysis for 2D

We can apply local mode analysis analogously to two-dimensional problems.

We represent the error as $e_{jm}^{(k)} = A(k)\exp(ij\theta_1 + im\theta_2)$ (mistake in book!)

where j and m run in x and y direction, and θ_1 and θ_2 represent waves in x and y direction.

Now we look for recurrence: $A(k+1) = G(\theta_1, \theta_2)A(k)$ where again

$G(\theta_1, \theta_2)$ is amplification factor corresponding to two wavenumbers.

The oscillatory modes are modes that are oscillatory in either one direction or both.

So smoothing factor is $\mu = \max_{\frac{\pi}{2} \le |\theta_j| \le \pi} |G(\theta_1, \theta_2)|$.

2D Example: Jacobi

Consider $-u_{xx} - u_{yy} + cu = f$

Discretization: $-u_{jm-1} - u_{j-1m} + (4 + h^2 c_{jm})u_{jm} - u_{j+1m} - u_{jm+1} = h^2 f_{jm}$

So weighted Jacobi iteration:

$$v_{jm}^{(k+1)} = \frac{\omega}{(4+h^2 c_{jm})}\left(h^2 f_{jm} + v_{jm-1}^{(k)} + v_{j-1m}^{(k)} + v_{j+1m}^{(k)} + v_{jm+1}^{(k)}\right) + (1-\omega)v_{jm}^{(k)}$$

For error this gives

$$e_{jm}^{(k+1)} = \frac{\omega}{(4+h^2 c_{jm})}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

For $c = 0$

$$e_{jm}^{(k+1)} = \frac{\omega}{4}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

2D Example: Jacobi

Analyze case $c = 0$

$$e_{jm}^{(k+1)} = \frac{\omega}{4}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

$$e_{jm}^{(k+1)} = \frac{\omega}{4}A(k)[\exp \iota[j\theta_1 + (m-1)\theta_2] + \exp \iota[j\theta_1 + (m+1)\theta_2] + \cdots] + (1-\omega)A\ldots =$$

$$\frac{\omega}{4}A(k)\exp \iota[j\theta_1 + m\theta_2](2\cos\theta_2 + 2\cos\theta_1) + (1-\omega)A(k)\exp \iota[j\theta_1 + m\theta_2] =$$

$$A(k)\exp \iota[j\theta_1 + m\theta_2](1 - \omega[1 - \tfrac{1}{4}(2\cos\theta_2 + 2\cos\theta_1)])$$

$$2\cos\theta_1 + 2\cos\theta_2 = 2\left(1 - 2\sin^2\left(\tfrac{\theta_1}{2}\right)\right) + 2\left(1 - 2\sin^2\left(\tfrac{\theta_2}{2}\right)\right) =$$

$$4 - 4\sin^2\left(\tfrac{\theta_1}{2}\right) - 4\sin^2\left(\tfrac{\theta_2}{2}\right)$$

$$A(k+1) = A(k)\left[1 - \omega\left(\sin^2\left(\tfrac{\theta_1}{2}\right) + \sin^2\left(\tfrac{\theta_2}{2}\right)\right)\right] \qquad \textbf{(mistake in book)}$$

$$G(\theta_1, \theta_2) = \left[1 - \omega\left(\sin^2\left(\tfrac{\theta_1}{2}\right) + \sin^2\left(\tfrac{\theta_2}{2}\right)\right)\right]$$

2D Example: Jacobi

Analyze case $c \neq 0$

$$e_{jm}^{(k+1)} = \frac{\omega}{4+h^2 c_{jm}}\left(e_{jm-1}^{(k)} + e_{j-1m}^{(k)} + e_{j+1m}^{(k)} + e_{jm+1}^{(k)}\right) + (1-\omega)e_{jm}^{(k)}$$

$$e_{jm}^{(k+1)} = A(k)\exp \iota[j\theta_1 + m\theta_2]\left(1-\omega + \frac{\omega}{4+h^2 c}(2\cos\theta_2 + 2\cos\theta_1)\right)$$

$$G(\theta_1, \theta_2) = \left(1-\omega + \frac{\omega}{4+h^2 c}(2\cos\theta_2 + 2\cos\theta_1)\right)$$

Alternatively

$$G(\theta_1, \theta_2) = \left(1-\omega + \frac{\omega}{4+h^2 c}(2\cos\theta_2 + 2\cos\theta_1)\right) =$$

$$1-\omega + \frac{\omega}{4}(2\cos\theta_2 + 2\cos\theta_1) - \frac{\omega}{4}(\cdots) + \frac{\omega}{4+h^2 c}(\cdots) =$$

$$G_0(\theta_1, \theta_2) + (2\cos\theta_2 + 2\cos\theta_1)\left(-\frac{\omega}{4} + \frac{\omega}{4+h^2 c}\right) =$$

$$G_0(\theta_1, \theta_2) - \frac{\omega}{4}\left(\frac{h^2 c}{4+h^2 c}\right)(2\cos\theta_2 + 2\cos\theta_1)$$

Convection-diffusion example

We solve several instances of the following equation

-u$_{xx}$+bu$_x$+cu=f

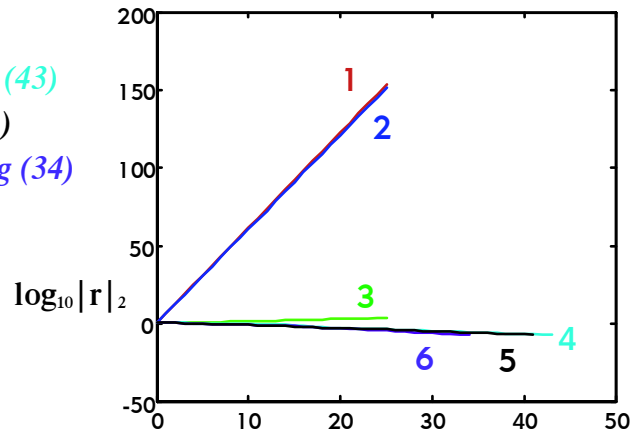**for various values of b, c, and h, and various choices in the multigrid algorithm.**

We show how smoothing analysis helps guide choices

Convection-diffusion example

$-u_{xx}+500u_x=0$, $h=1/64$, $nrel=10$,
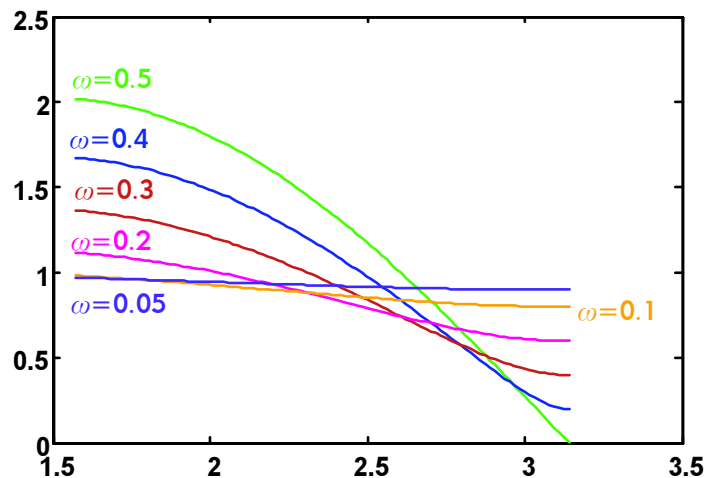
1. $\omega=0.5$, injection
2. $\omega=0.5$, full weighting
3. $\omega=0.1$, injection
4. $\omega=0.1$, full weighting (43)
5. $\omega=0.05$, injection (41)
6. $\omega=0.05$, full weighting (34)

Convection-diffusion example
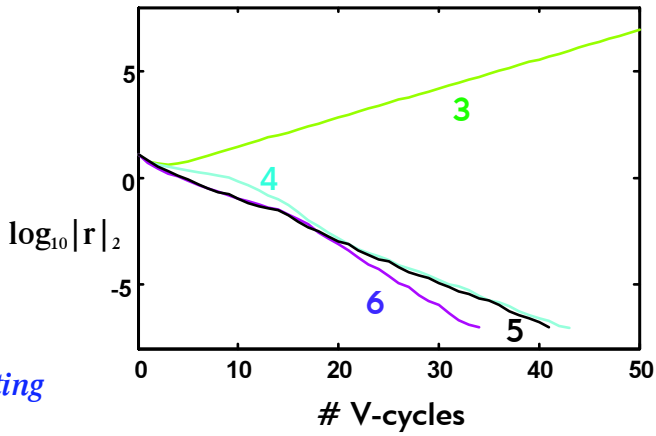
Amplification factors for the oscillatitory modes and various weights

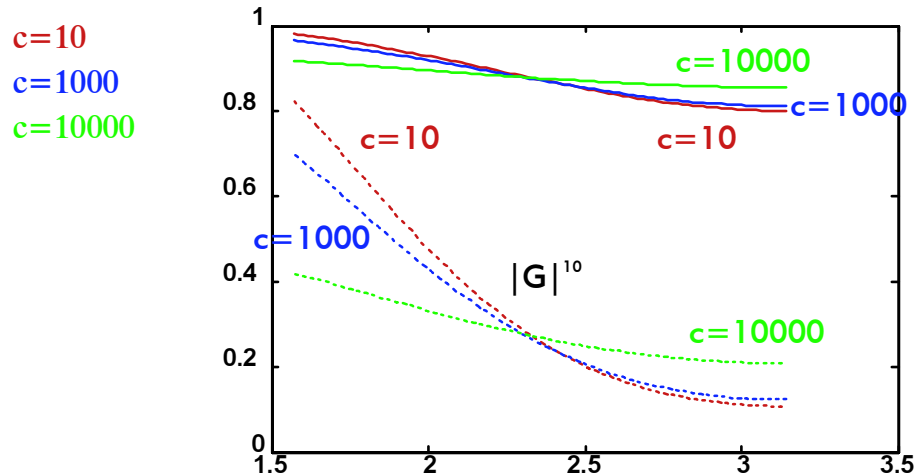# Convection-diffusion example

$\log_{10}|\mathbf{r}|_2$

V-cycles

1. *w=0.5, injection*
2. *w=0.5, full weighting*
3. *w=0.1, injection*
4. *w=0.1, full weighting (43)*
5. *w=0.05, injection (41)*
6. *w=0.05, full weighting (34)*

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Convection-diffusion example

**Amplification factors for the oscillatitory modes,
$\omega=0.1$, injection, and various values for c**

c=10
c=1000
c=10000

$|G|^{10}$

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