

(TD)DFT for noncollinear spins: orbital functionals, semilocal approximations, and xc torques

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DFT for noncollinear magnetism:

- ▶ XC orbital functionals and new meta-GGA
- ▶ application to Cr_3 and Cr_5

Significance of XC magnetic torques:

- ▶ Spin dynamics in Hubbard clusters with spin frustration

C. A. Ullrich, Phys. Rev. B **98**, 035140 (2018)

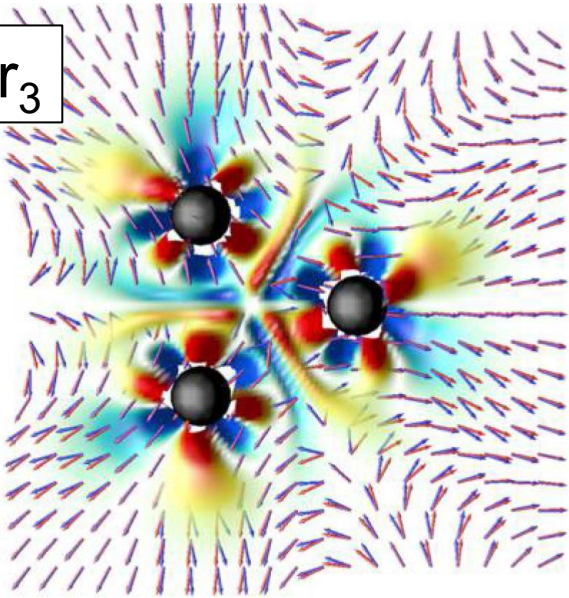
C. A. Ullrich, Phys. Rev. A **100**, 012516 (2019)

E. A. Pluhar and C. A. Ullrich, Phys. Rev. B **100**, 125135 (2019)

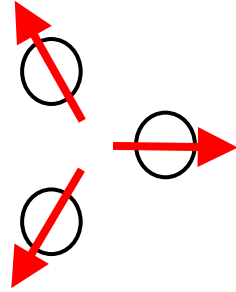
N. Tancogne-Dejean, A. Rubio, and C. A. Ullrich, Phys. Rev. B **107**, 165111 (2023)

D. Hill, J. Shotton, and C. A. Ullrich, Phys. Rev. B **107**, 115134 (2023)

Cr₃

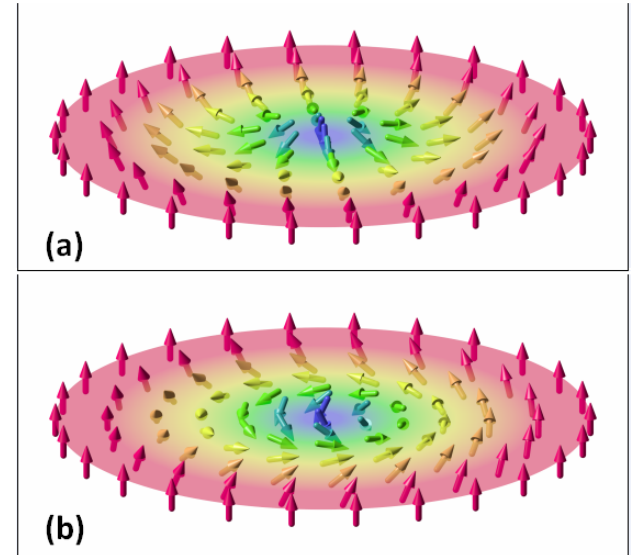


G. Scalmani and M.J. Frisch, JCTC **8**, 2193 (2012)



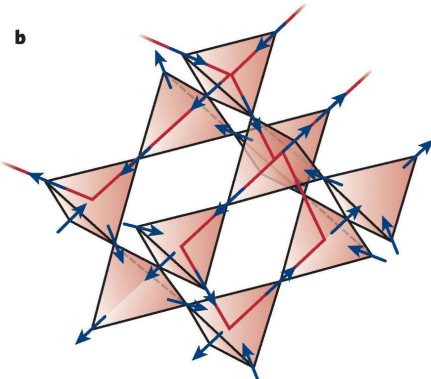
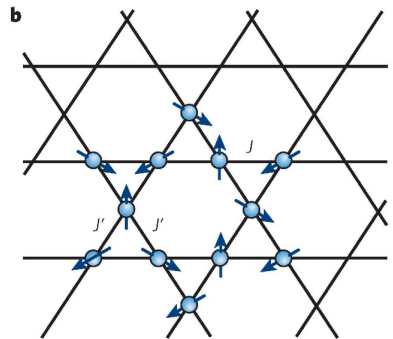
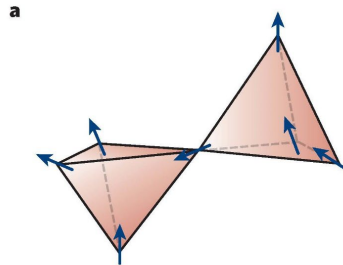
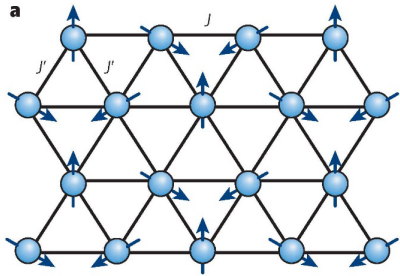
Romming et al., Science **341**, 636 (2013)

skyrmions



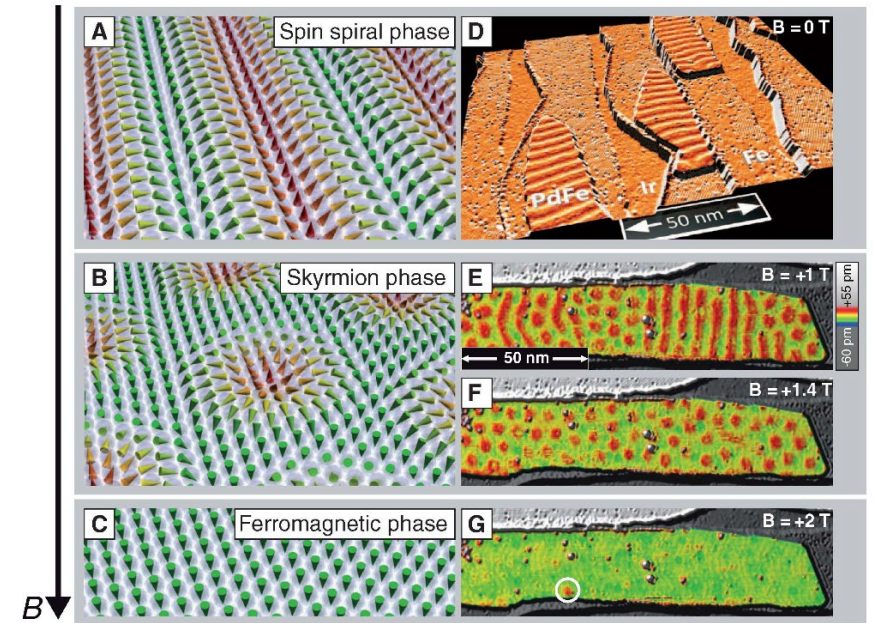
https://en.wikipedia.org/wiki/Magnetic_skyrmion

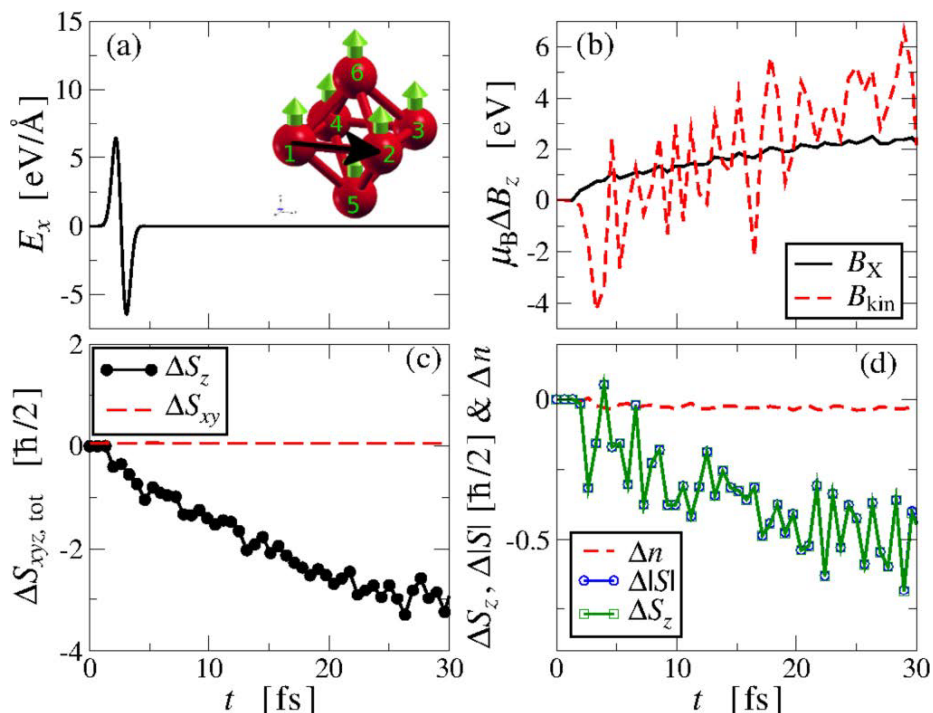
Many noncollinear magnetic materials exist in nature



spin frustrations and quantum spin liquids

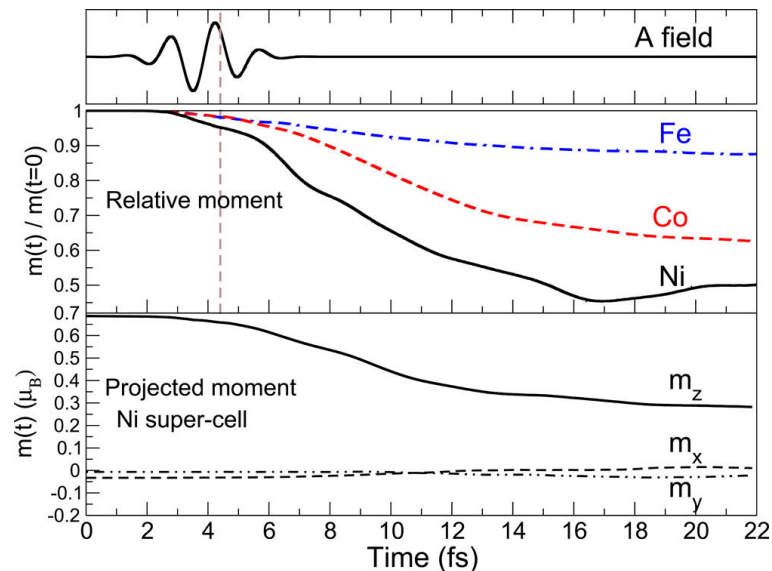
L. Balents, Nature **464**, 199 (2010)



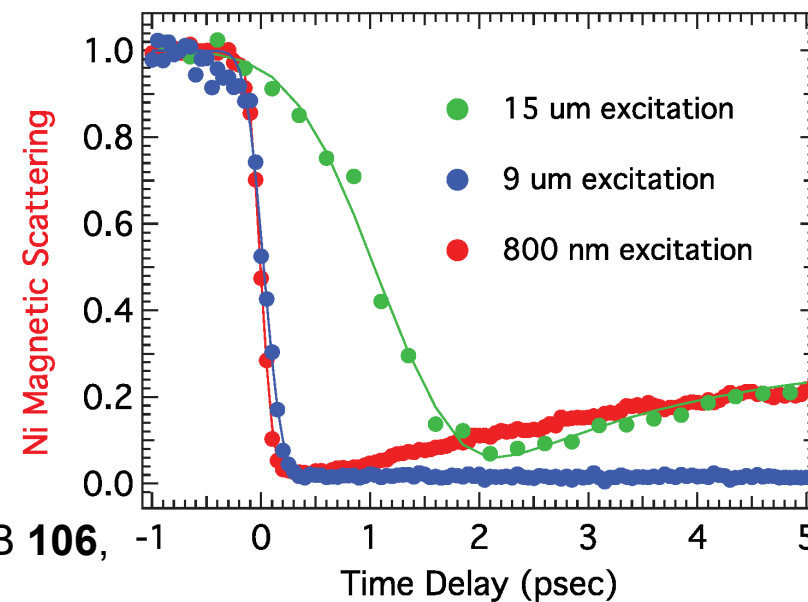
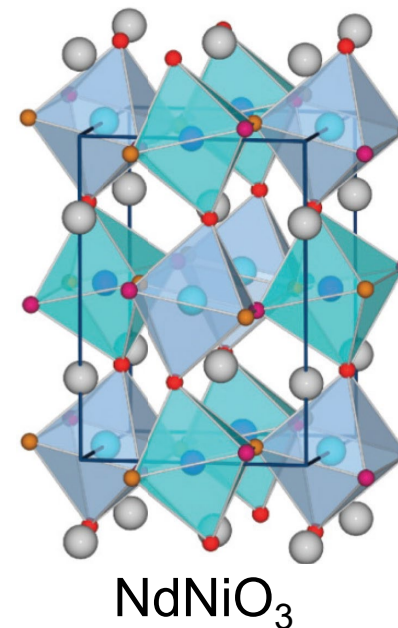


Simoni, Stamenova & Sanvito,
PRB **95**, 024412 (2017)

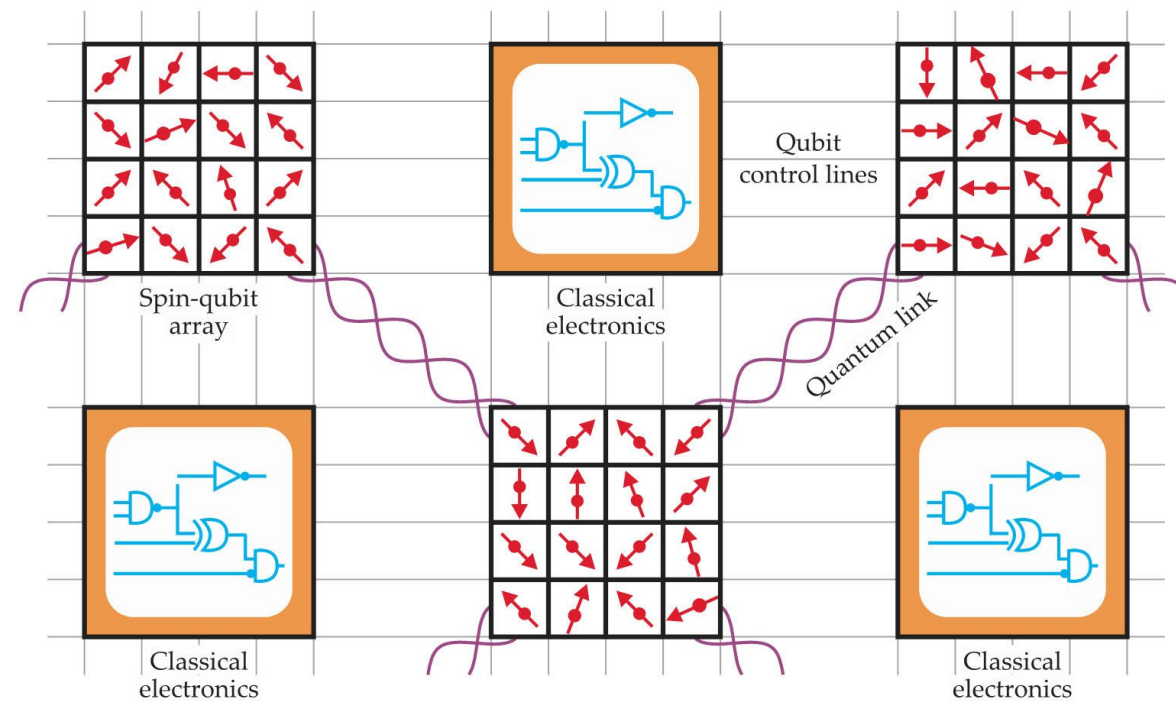
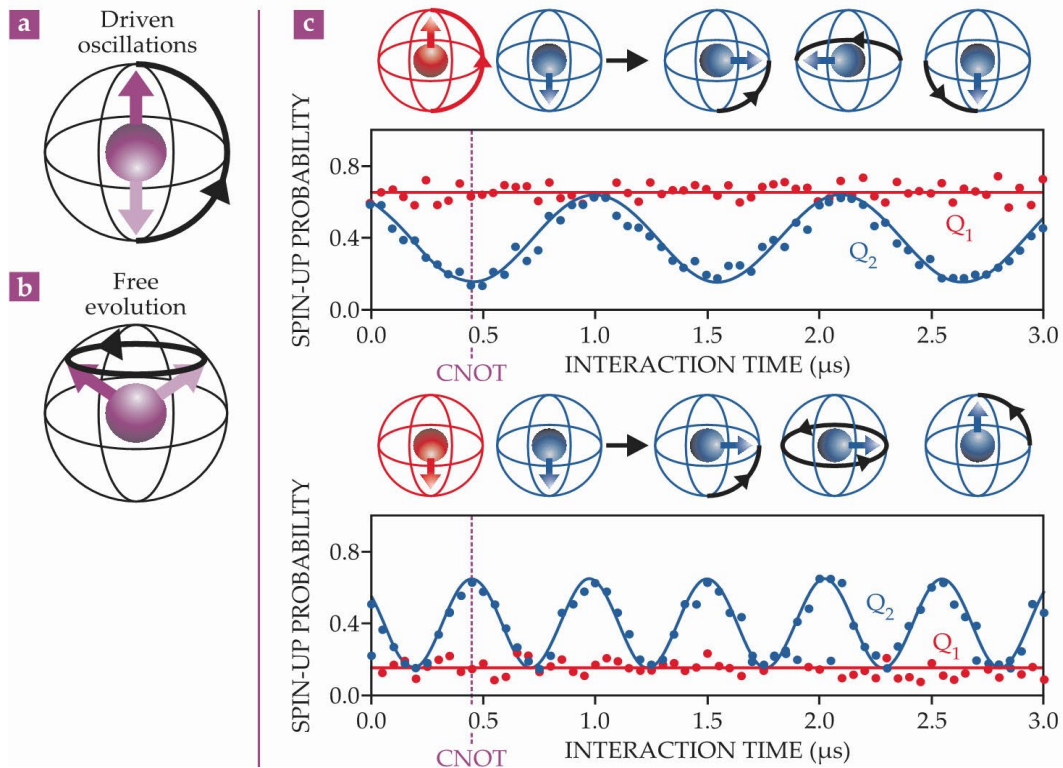
Magnetic materials can be controlled by fs laser pulses



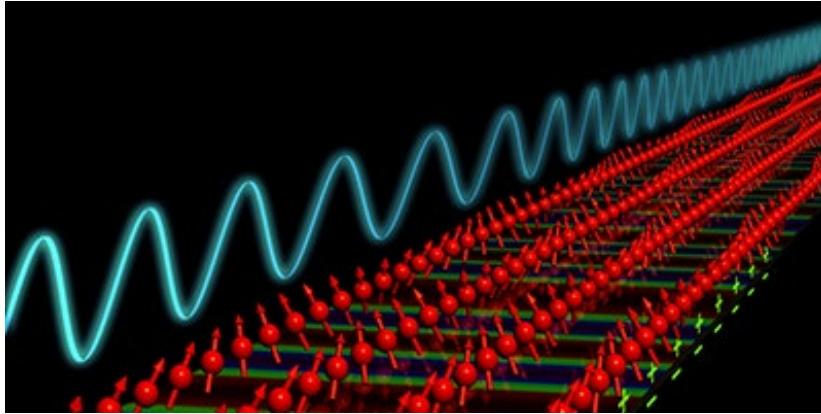
Krieger et al., JCTC **11**,
4870 (2015)



Stoica et al., PRB **106**,
165104 (2022)

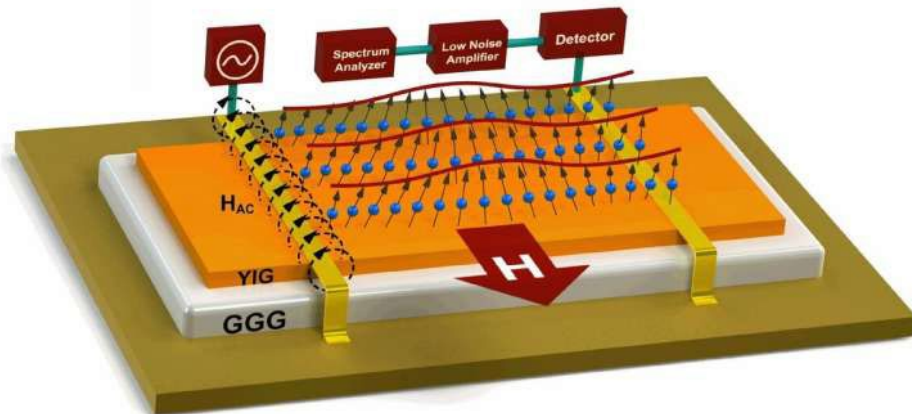


Vandersypen & Eriksson, *Physics Today* **72**, 38-45 (2019)

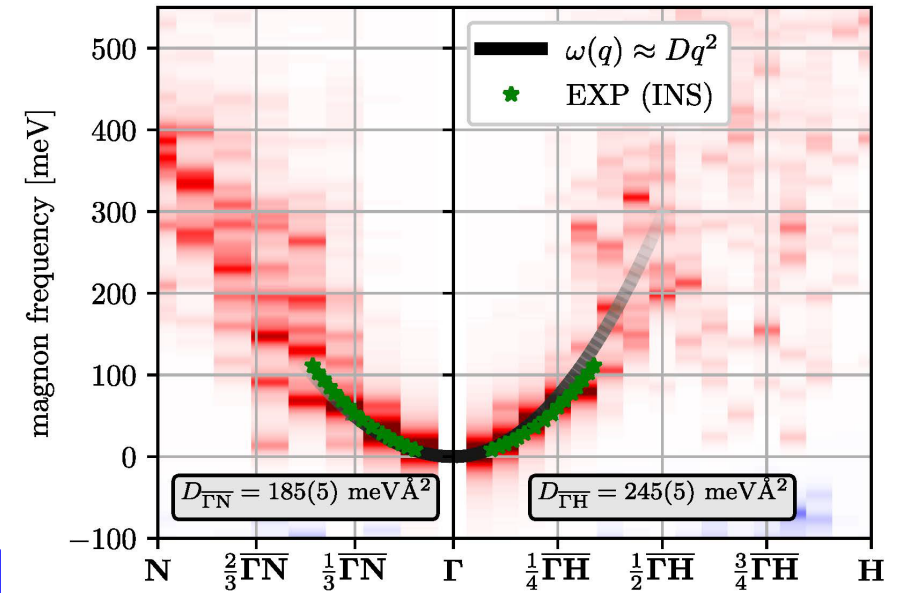


X. Zhang, T. Liu, M. E. Flatté, and H. X. Tang
 Phys. Rev. Lett. **113**, 037202 (2014)

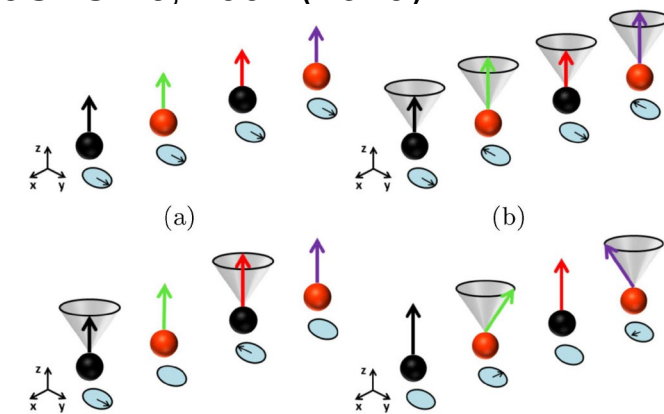
- ▶ Spin waves as carrier of information
- ▶ Need materials with minimal losses
- ▶ LR or RT-TDDFT for magnons



<https://phys.org/news/2019-03-magnonic-devices-electronics-noise.html>



N. Tancogne-Dejean, F.G. Eich & A. Rubio,
 JCTC **16**, 1007 (2020)



N. Singh, P. Elliott, J.K. Dewhurst & S. Sharma,
 PRB **103**, 134402 (2021)

N -electron system in a magnetic field, acting on spins only:

$$\hat{H} = \sum_{j=1}^N \left(-\frac{\nabla_j^2}{2} + V(\mathbf{r}_j) + \boldsymbol{\sigma}_j \cdot \mathbf{B}(\mathbf{r}_j) \right) + \frac{1}{2} \sum_{j \neq k}^N \frac{1}{|\mathbf{r}_j - \mathbf{r}_k|}$$

(\mathbf{B} contains μ_B)

U. von Barth and L. Hedin, J. Phys. C **5**, 1629 (1972)

O. Gunnarsson and B. I. Lundqvist, PRB **13**, 4274 (1976)

N. I. Gidopoulos, PRB **75**, 134408 (2007)

spin-density matrix:

$$\underline{\underline{n}}(\mathbf{r}) = \langle \Psi | \hat{\psi}_\beta^\dagger(\mathbf{r}) \hat{\psi}_\alpha(\mathbf{r}) | \Psi \rangle = \begin{pmatrix} n_{\uparrow\uparrow} & n_{\uparrow\downarrow} \\ n_{\downarrow\uparrow} & n_{\downarrow\downarrow} \end{pmatrix}$$

density: $n(\mathbf{r}) = n_{\uparrow\uparrow}(\mathbf{r}) + n_{\downarrow\downarrow}(\mathbf{r})$

magnetization: $\mathbf{m}(\mathbf{r}) = \text{tr} \{ \boldsymbol{\sigma} \underline{\underline{n}}(\mathbf{r}) \}$

$$\mathbf{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} n_{\uparrow\downarrow} + n_{\downarrow\uparrow} \\ i(n_{\uparrow\downarrow} - n_{\downarrow\uparrow}) \\ n_{\uparrow\uparrow} - n_{\downarrow\downarrow} \end{pmatrix}$$

2-component
spinors:

$$\Psi_i(\mathbf{r}) = \begin{pmatrix} \psi_{i\uparrow}(\mathbf{r}) \\ \psi_{i\downarrow}(\mathbf{r}) \end{pmatrix}$$

v. Barth & Hedin (1972)
Gunnarsson & Lundqvist (1976)

$$\left[\left(-\frac{\nabla^2}{2} + V_{\text{ext}+\text{H}+\text{xc}}(\mathbf{r}) \right) I + \boldsymbol{\sigma} \cdot \mathbf{B}_{\text{ext}+\text{xc}}(\mathbf{r}) \right] \Psi_i(\mathbf{r}) = \varepsilon_i \Psi_i(\mathbf{r})$$

$$V_{\text{xc}}(\mathbf{r}) = \frac{\delta E_{\text{xc}}[n, \mathbf{m}]}{\delta n(\mathbf{r})}$$

$$\mathbf{B}_{\text{xc}}(\mathbf{r}) = \frac{\delta E_{\text{xc}}[n, \mathbf{m}]}{\delta \mathbf{m}(\mathbf{r})}$$

local xc torque:

$$\boldsymbol{\tau}_{\text{xc}} = \mathbf{m} \times \mathbf{B}_{\text{xc}}$$

Capelle, Vignale & Györfy, PRL **87**, 206403 (2001)

Time evolution of magnetization
in many-body system:

$$\frac{d\mathbf{m}(\mathbf{r}, t)}{dt} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = \mathbf{m}(\mathbf{r}, t) \times \mathbf{B}_{\text{ext}}(\mathbf{r}, t)$$

Formally equivalent:

$$\frac{d\mathbf{m}(\mathbf{r}, t)}{dt} + \nabla \cdot \mathbf{J}_{KS}(\mathbf{r}, t) = \mathbf{m}(\mathbf{r}, t) \times \mathbf{B}_{\text{ext}+\text{xc}}(\mathbf{r}, t)$$

Zero-torque theorem:

$$\int d\mathbf{r} \mathbf{m}(\mathbf{r}, t) \times \mathbf{B}_{\text{xc}}(\mathbf{r}, t) = 0$$

Some approximations may violate ZTT, but
the ZTT can be enforced as constraint.
Pluhar and Ullrich, Phys. Rev. B **100**, 125135 (2019)

Assume local spin quantization axis (Kübler 1988, Sandratskii 1998):

$$\begin{aligned} \mathbf{B}_{\text{xc}}^{\text{LSDA}}(\mathbf{r}) &= \frac{\delta E_{\text{xc}}^{\text{LSDA}}[n, \mathbf{m}]}{\delta \mathbf{m}(\mathbf{r})} \\ &\equiv \frac{\partial e_{\text{xc}}^{\text{unif}}(\bar{n}, \bar{m})}{\partial \bar{m}} \bigg|_{\substack{\bar{n}=n(\mathbf{r}) \\ \bar{m}=m(\mathbf{r})}} \frac{n(\mathbf{r})\mathbf{m}(\mathbf{r})}{m(\mathbf{r})} \end{aligned}$$

$\mathbf{B}_{\text{xc}}^{\text{LSDA}}(\mathbf{r})$ is always parallel to $\mathbf{m}(\mathbf{r})$

$$\Rightarrow \boldsymbol{\tau}_{\text{xc}}^{\text{LSDA}} = 0$$

► Spin-spiral reference state of homogeneous electron gas

M.I. Katsnelson and V.P. Antropov, PRB **67**, 140406 (2003)

F.G. Eich and E.K.U. Gross, PRL **111**, 156401 (2013)

► GGAs for noncollinear spin systems

Scalmani & Frisch, JCTC **8**, 2193 (2012)

Bulik, Scalmani, Frisch & Scuseria, PRB **87**, 035117 (2013)

Pu et al., Phys. Rev. Res. **5**, 013036 (2023)

Eich, Pittalis & Vignale, PRB **88**, 245102 (2013)

Pittalis, Vignale & Eich, PRB **96**, 035141 (2017)

Tancogne-Dejean, Rubio & Ullrich, PRB **107**, 165111 (2023)

} upgrade existing GGAs

} new (m)GGAs

► Source-free LSDA

Sharma, Gross, Sanna & Dewhurst, JCTC **14**, 1247 (2018)

Dewhurst, Sanna & Sharma, EPJB **91**, 218 (2018)

► Orbital functionals:

- no reference system (electron gas) needed
- works for Hubbard model and real materials alike
- will produce xc torques

$$E_x = -\frac{1}{2} \int \int \frac{d\mathbf{r}d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \text{Tr} \left[\underline{\underline{\gamma}}(\mathbf{r}, \mathbf{r}') \underline{\underline{\gamma}}(\mathbf{r}', \mathbf{r}) \right]$$

Can construct local EXX potential via OEP or KLI

Krieger, Li & Iafrate, PRA **45**, 101 (1992)

Sharma et al., PRL **98**, 196405 (2007)

Noncollinear Slater potential:

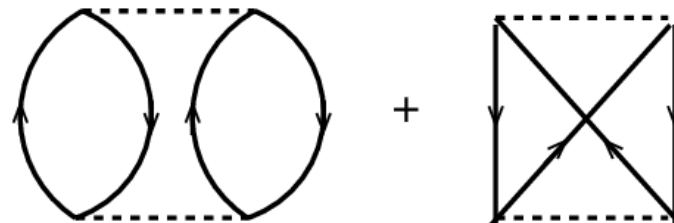
$$\underline{\underline{V}}_x^S(\mathbf{r}) \underline{\underline{n}}(\mathbf{r}) + \underline{\underline{n}}(\mathbf{r}) \underline{\underline{V}}_x^S(\mathbf{r}) = -2 \int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \underline{\underline{\gamma}}(\mathbf{r}, \mathbf{r}') \underline{\underline{\gamma}}(\mathbf{r}', \mathbf{r})$$

C.A. Ullrich, PRB **98**, 035140 (2018)

► **Self-interaction correction (Perdew-Zunger):**

$$E_{\text{xc}}^{\text{SIC}}[n, \mathbf{m}] = E_{\text{xc}}^{\text{approx}}[n, \mathbf{m}] - \sum_j \left(E_{\text{H}}[n_j] + E_{\text{xc}}^{\text{approx}}[n_j, \mathbf{m}_j] \right)$$

► **Many-body perturbation theory (MP2 or GL2)**



► **Functionals based on 2-body density matrix**

$$E_{\text{c}} = W[\underset{=2}{\gamma}] - E_{\text{H}}[n] - E_{\text{x}}[\underset{=}{\gamma}]$$

Colle & Salvetti (1975)

► **STLS: based on fluctuation-dissipation theorem**

Singwi, Sjölander, Tosi & Land, PR **176**, 589 (1968)
 C. A. Ullrich, Phys. Rev. B **98**, 035140 (2018)

S. Pittalis, G. Vignale & F. G. Eich, Phys. Rev. B **96**, 035141 (2017)

N. Tancogne-Dejean, A. Rubio, and C. A. Ullrich, Phys. Rev. B **107**, 165111 (2023)

Exact U(1)xSU(2) invariant exchange energy:

$$E_x = -\frac{1}{2} \int \int \frac{d\mathbf{r}d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \text{Tr} \left[\underline{\underline{\gamma(\mathbf{r}, \mathbf{r}')}} \underline{\underline{\gamma(\mathbf{r}', \mathbf{r})}} \right]$$

Rewrite this in terms of exchange hole:

$$E_x = -\frac{1}{2} \int d\mathbf{r} n(\mathbf{r}) \int d\mathbf{r}' \frac{h_x(\mathbf{r}, \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \quad h_x(\mathbf{r}, \mathbf{r}') = \frac{\text{Tr} \left[\underline{\underline{\gamma(\mathbf{r}, \mathbf{r}')}} \underline{\underline{\gamma(\mathbf{r}', \mathbf{r})}} \right]}{n(\mathbf{r})}$$

Idea: • follow Becke and Roussel, PRA **39**, 3761 (1989)

- short-range expansion of spherical average of exchange hole
- gauge invariant, recovers collinear limit and homogeneous limit

Spherical average of effective exchange hole ($s = \text{radius}$):

$$h_x(s) = n\zeta_x + s^2 k_F^2 \left[\frac{2}{3} q - \frac{\gamma}{5} \alpha \right]$$

$$\zeta_x(\mathbf{r}) = \frac{1}{2} \left(1 + \frac{|\mathbf{m}(\mathbf{r})|^2}{n(\mathbf{r})^2} \right)$$

spin polarization factor

$$\alpha = (\bar{\tau} - \tau^W) / \bar{\tau}^{\text{unif}}$$

iso-orbital indicator

$$q = \nabla^2 n / 4k_F^2 n \zeta_x$$

reduced gradient

Gauge invariant kinetic energy density:

$$2n\bar{\tau} = \text{Tr} \left[\underline{\underline{n\tau}} + \underline{\underline{\tau n}} - 2i\nabla(\underline{\underline{nj}} + \underline{\underline{jn}}) - 2\underline{\underline{\mathbf{j} \cdot \mathbf{j}}} \right] \\ + \sum_{\sigma} \left[n_{\sigma\sigma} \nabla^2 n_{\bar{\sigma}\bar{\sigma}} - \Re(n_{\sigma\bar{\sigma}} \nabla^2 n_{\bar{\sigma}\sigma}) - \frac{1}{2} |\nabla n_{\sigma\bar{\sigma}}|^2 + \frac{1}{2} \nabla n_{\sigma\sigma} \cdot \nabla n_{\bar{\sigma}\bar{\sigma}} \right]$$

Fitting the averaged exchange hole with a hydrogenic model:

$$E_x = -3 \frac{(3\pi^2)^{1/3}}{4\pi} \int dr n(\mathbf{r})^{4/3} \zeta_x(\mathbf{r})^{1/3} F_x(\mathbf{r})$$

$$F_x(\mathbf{r}) = \frac{4\pi^{2/3} e^{x(\mathbf{r})/3}}{3^{4/3} x(\mathbf{r})} \left[1 - e^{-x(\mathbf{r})} \left(1 - \frac{x(\mathbf{r})}{2} \right) \right]$$

enhancement factor (x comes from the hydrogenic fitting of h_x , see Becke-Roussel)

S. Pittalis, G. Vignale & F. G. Eich, Phys. Rev. B **96**, 035141 (2017)

Colle & Salvetti (1975):

$$E_c = -4a \int d\mathbf{r} \frac{\rho_2(\mathbf{r}, \mathbf{r})}{n(\mathbf{r})} \left[\frac{1 + br_s^8(\mathbf{r}) [\nabla_s^2 \rho_2(\mathbf{r}, \mathbf{s})]_{s=0} e^{-cr_s(\mathbf{r})}}{1 + dr_s(\mathbf{r})} \right]$$

approximate the 2-body RDM as

$$\rho_2^{HF}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2} \left[n(\mathbf{r}_1)n(\mathbf{r}_2) - \text{Tr}[\underline{\underline{\gamma}}(\mathbf{r}_1, \mathbf{r}_2)\underline{\underline{\gamma}}(\mathbf{r}_2, \mathbf{r}_2)] \right]$$

Then follow the derivation by Lee, Yang & Parr (LYP) (1988)

R. Colle and O. Salvetti, *Theor. Chim. Acta* **37**, 329 (1975)

C. Lee, W. Yang, and R. G. Parr, *Phys. Rev. B* **37**, 785 (1988)

$$E_c = -a \int d\mathbf{r} n \zeta_c^2 \left[\frac{1 + (br_s^5 / 2) \left[\bar{\tau} + (|\nabla \mathbf{m}| / 4n) - 3\tau^W \right] e^{-cr_s}}{1 + dr_s} \right]$$

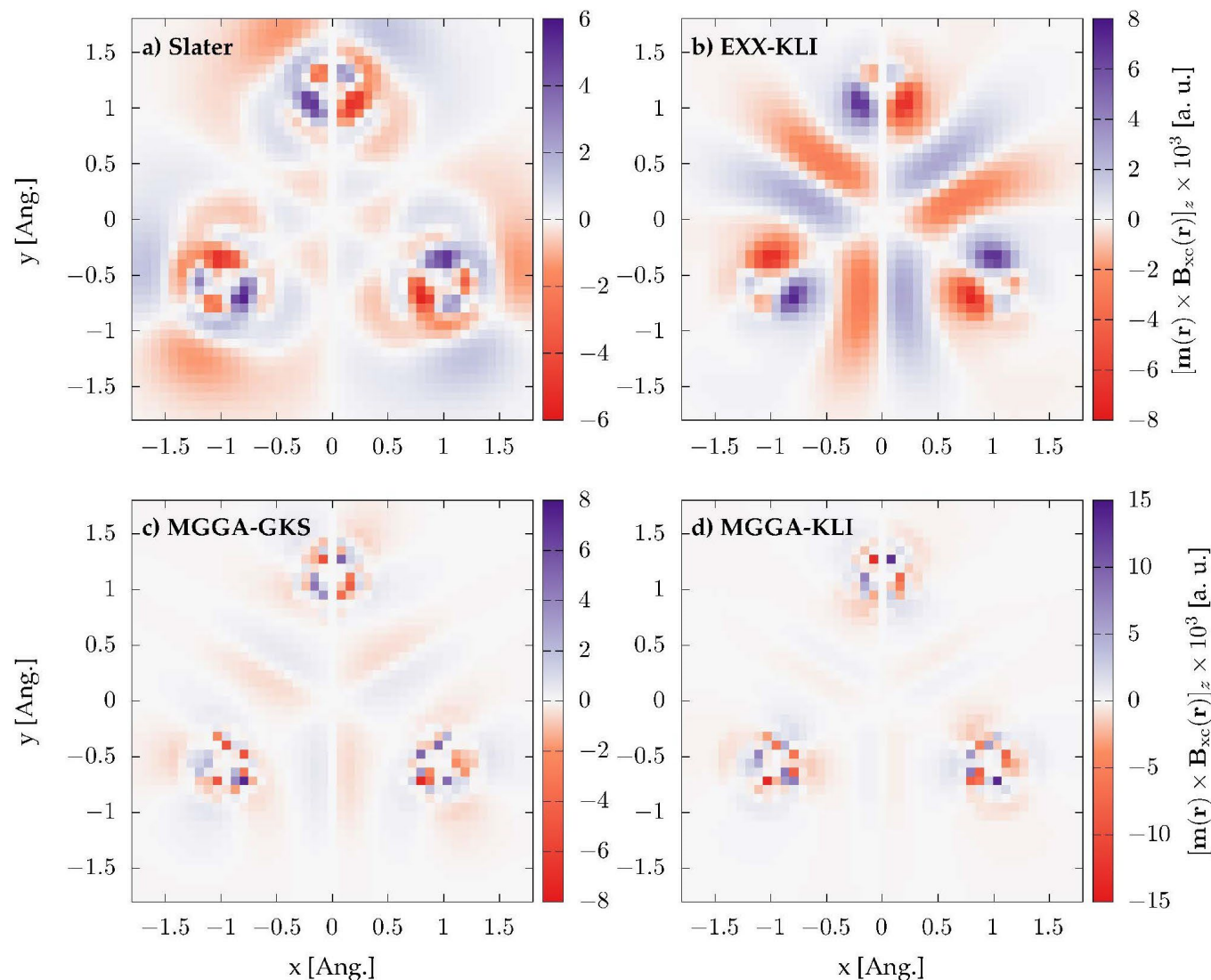
$$\zeta_c(\mathbf{r}) = 1 - \frac{|\mathbf{m}(\mathbf{r})|^2}{n(\mathbf{r})^2}$$

rescaled LYP parameters:

$$a = 0.04918 \quad b = 0.132(4\pi/3)^{8/3}$$

$$c = 0.2533(4\pi/3)^{1/3} \quad d = 0.349(4\pi/3)^{1/3}$$

Note: to convert this into a GGA requires 2nd order gradient expansion of $\bar{\tau}$ which must preserve gauge invariance (future work)



Real-space calculation with Octopus

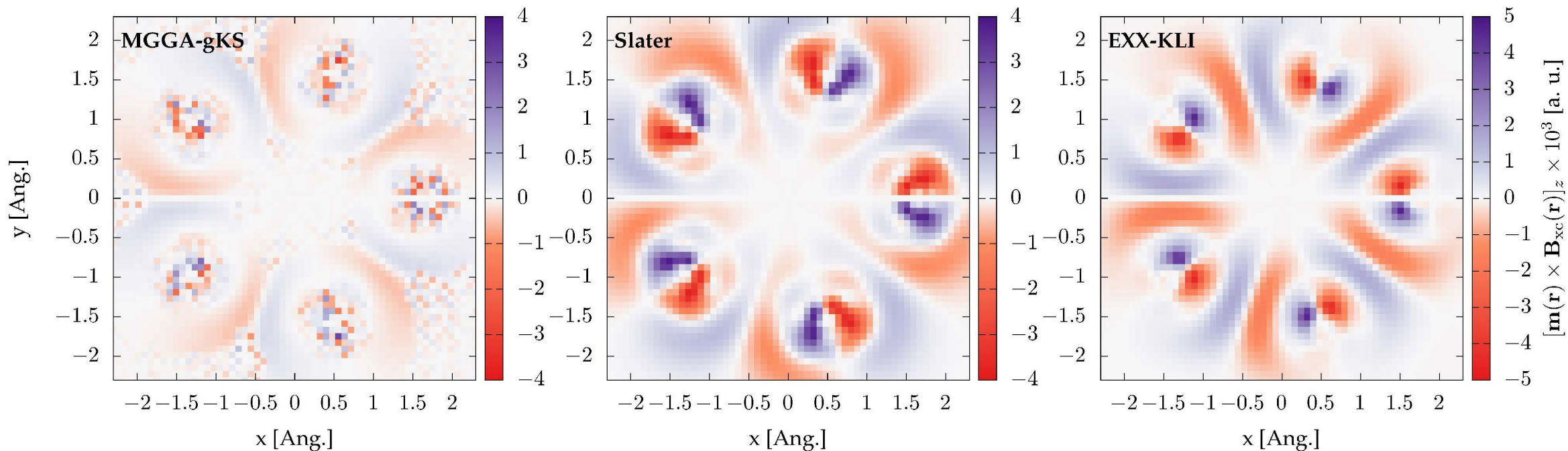
Grid spacing 0.1 Bohr

HGH fully relativistic pseudopotentials

Similar magnitude as Slater and EXX
Similar shape around the atoms

Improved magnetic moments and
ionization energy compared to LSDAx

Sign not correct in the interstitial region:
failure of the short-range expansion
Path toward further improvements

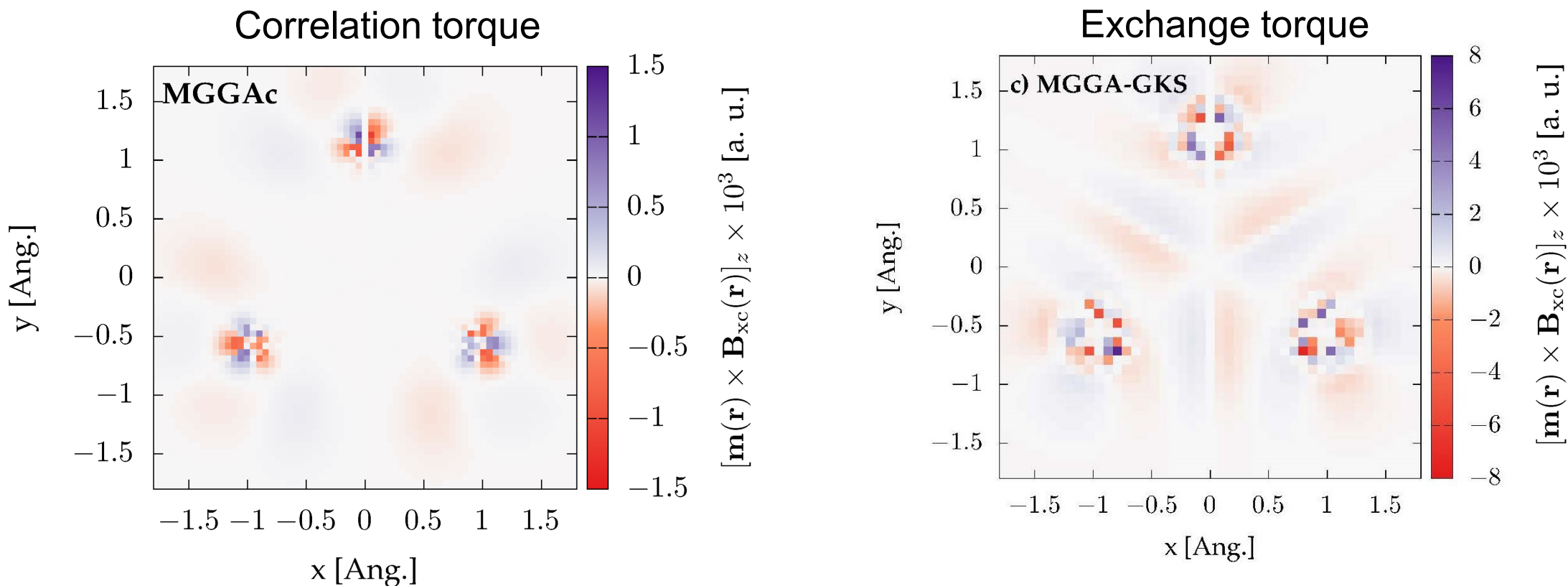


Same conclusions as for Cr₃:

- Similar magnitude as Slater and EXX
- Similar shape around the atoms
- Improved magnetic moments and ionization energy compared to LSDAx
- Sign not correct in the interstitial region

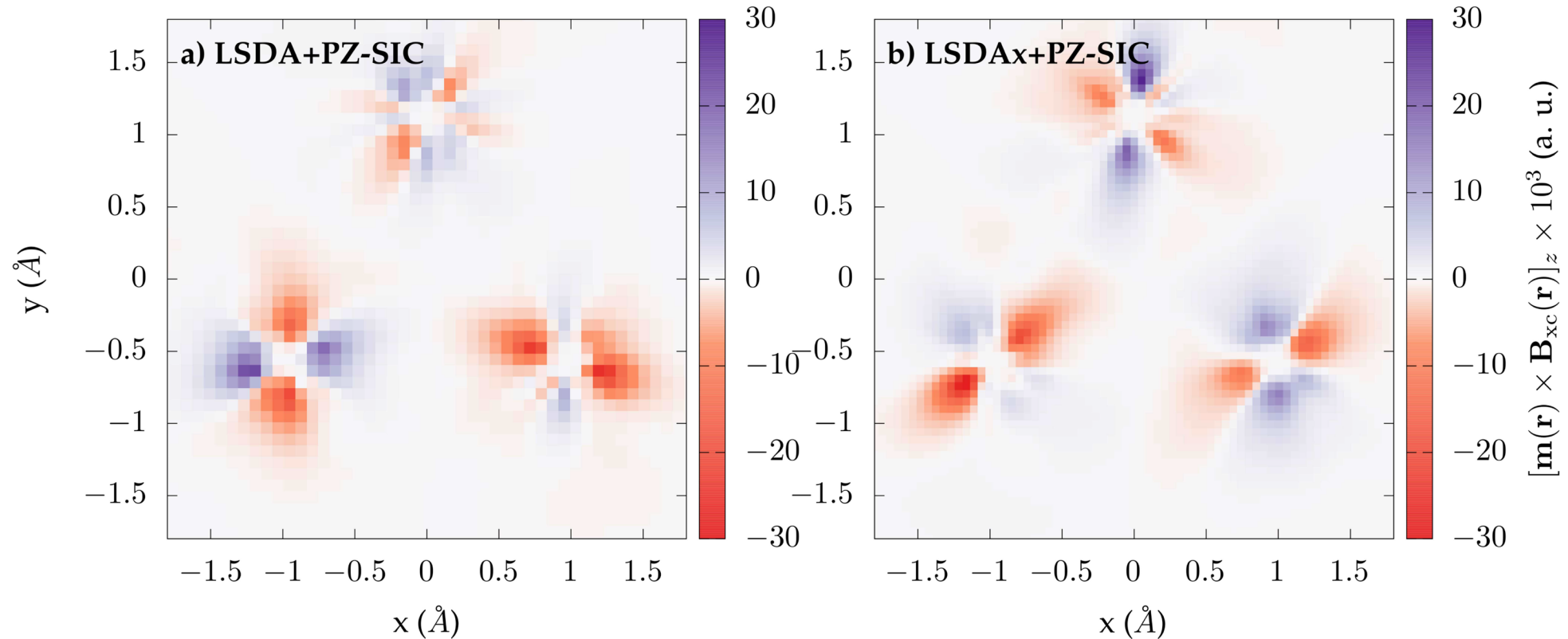
TABLE I. Local magnetic moment $|\mathbf{m}|$, in μ_B , and ionization potential I_p , in eV, of the Cr atoms in Cr_3 obtained for different levels of theory (see text).

Functional	$ \mathbf{m} $	I_p
LSDA	1.67	2.90
LSDAx	2.66	2.30
LSDAx+MGGA _c -gKS	1.81	2.60
MGGA _x +MGGA _c -gKS ($\gamma = 0.8$)	2.30	4.61
MGGA _x -gKS ($\gamma = 0.8$)	3.04	3.65
MGGA _x -gKS ($\gamma = 1$)	3.07	3.53
MGGA _x -KLI ($\gamma = 0.8$)	3.09	3.59
MGGA _x -KLI ($\gamma = 1$)	3.14	3.47
Slater	3.48	6.52
EXX-KLI	3.81	4.68
Hartree-Fock	3.86	4.86



- Correlation torque:
- smaller magnitude,
 - more localized around nuclei
 - opposite sign of exchange torque

} intricate interplay between X and C torques



- ▶ Large difference between LSDA-SIC and LSDAx-SIC
- ▶ quite different from EXX (torques seem a bit high)
- ▶ zero-torque theorem?

1. How good are approximate xc functionals compared to exact benchmark solutions?
2. What is the significance of xc torques for noncollinear spins and their dynamics?

Small Hubbard clusters are ideal model systems to study this!

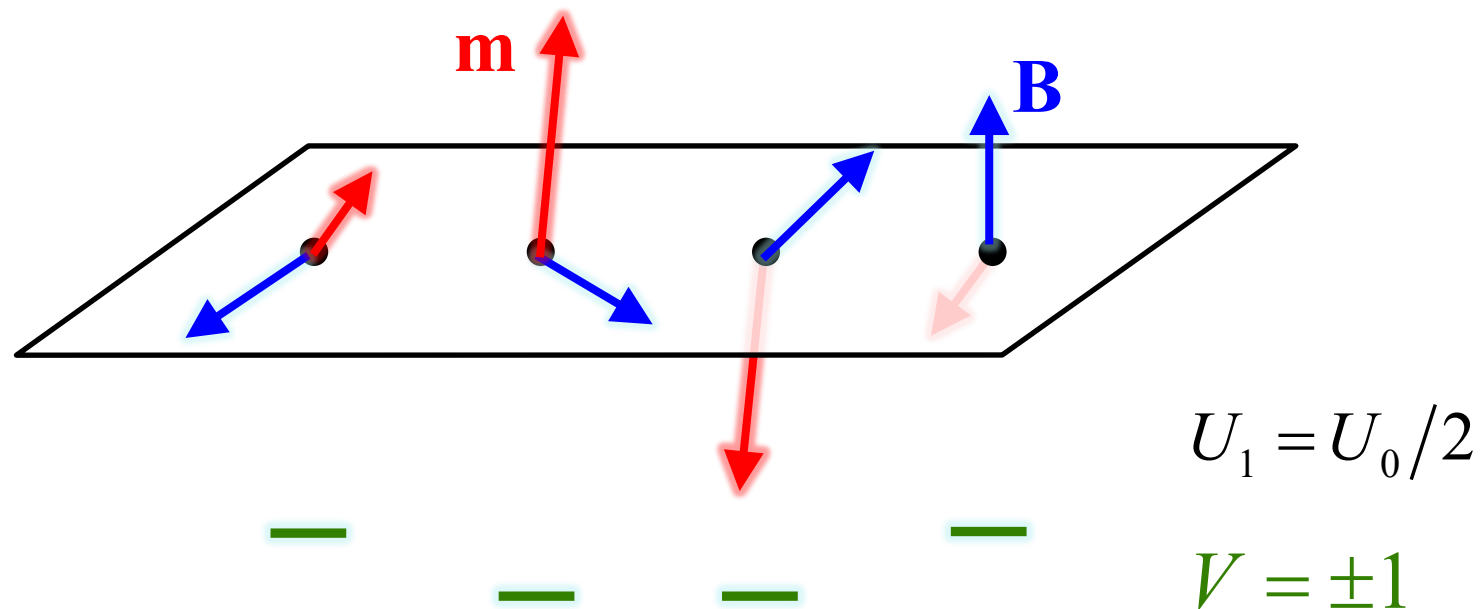
$$\begin{aligned}
 \hat{H} = & -t \sum_{\langle k,l \rangle \sigma} [\hat{c}_{k\sigma}^+ \hat{c}_{l\sigma} + \hat{c}_{l\sigma}^+ \hat{c}_{k\sigma}] + U_0 \sum_k \hat{c}_{k\uparrow}^+ \hat{c}_{k\uparrow} \hat{c}_{k\downarrow}^+ \hat{c}_{k\downarrow} \\
 & + U_1 \sum_{\langle k,l \rangle} (\mathbf{c}_k^\dagger \mathbf{c}_k)(\mathbf{c}_l^\dagger \mathbf{c}_l) + \sum_k \left[V_k \mathbf{c}_k^\dagger \mathbf{c}_k + \mathbf{B}_k \cdot \mathbf{c}_k^\dagger \boldsymbol{\sigma} \mathbf{c}_k \right]
 \end{aligned}$$

Why include nearest-neighbor interaction U_1 ?

Using only U_0 exchange becomes trivial:

$$V_{x,k} = -U_0 n_k$$

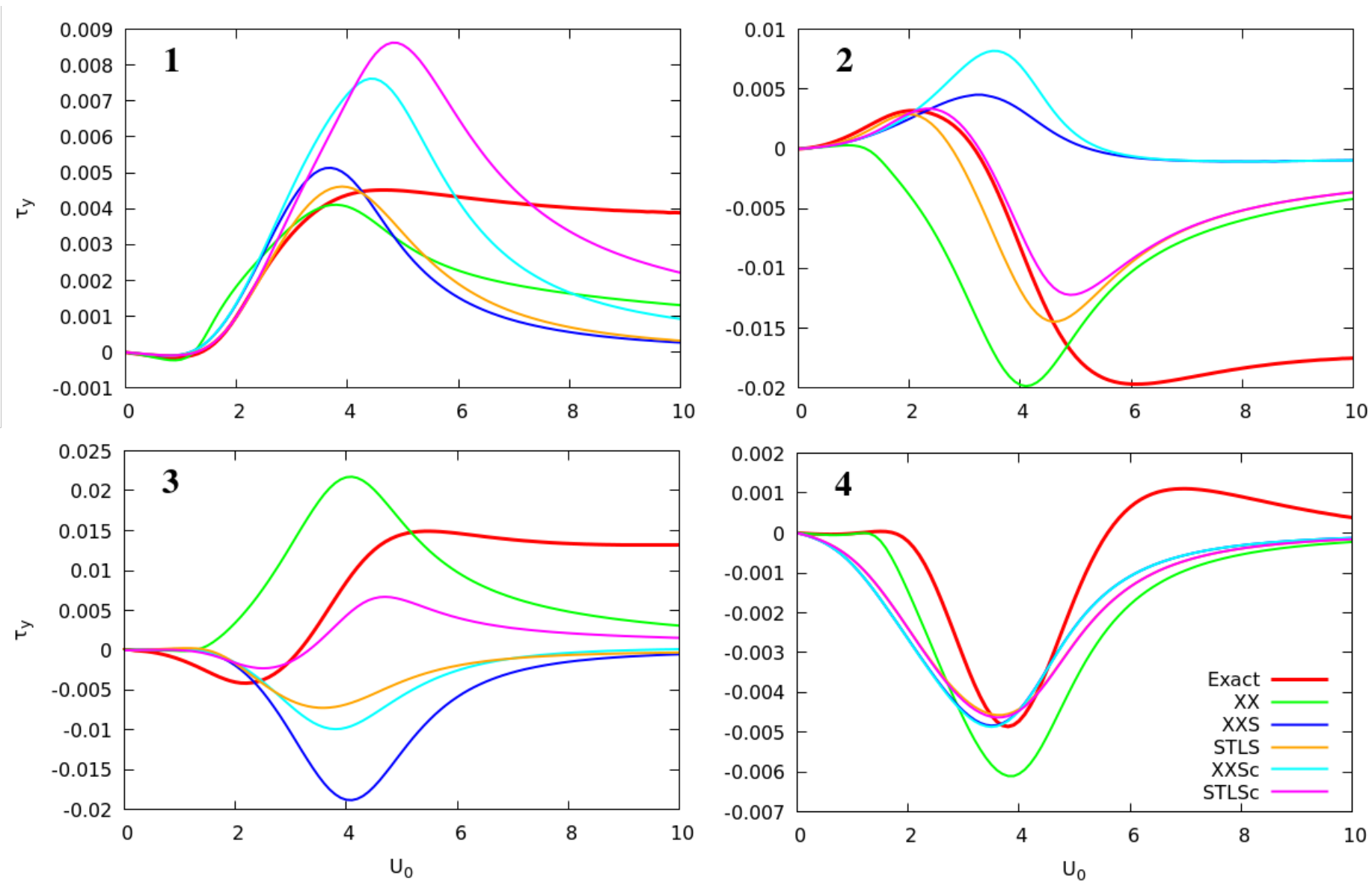
$$\mathbf{B}_{x,k} = -U_0 \mathbf{m}_k \quad \Rightarrow \quad \text{no x-torque!}$$



- ▶ Hubbard model, only U_0 : xc torque is **purely correlation**
 C.A. Ullrich, PRB **98**, 035140 (2018)
- ▶ NN interaction U_1 : xc torque can have **exchange contributions**
 (first fully nontrivial case: tetramer)
 E. A. Pluhar and C. A. Ullrich, Phys. Rev. B **100**, 125135 (2019)



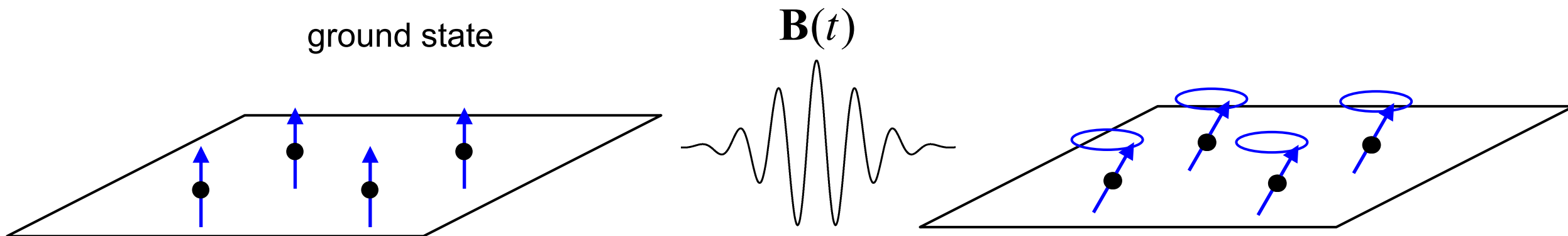
Exact vs approximate xc torques



Approximations are reasonable for moderate interaction strengths.

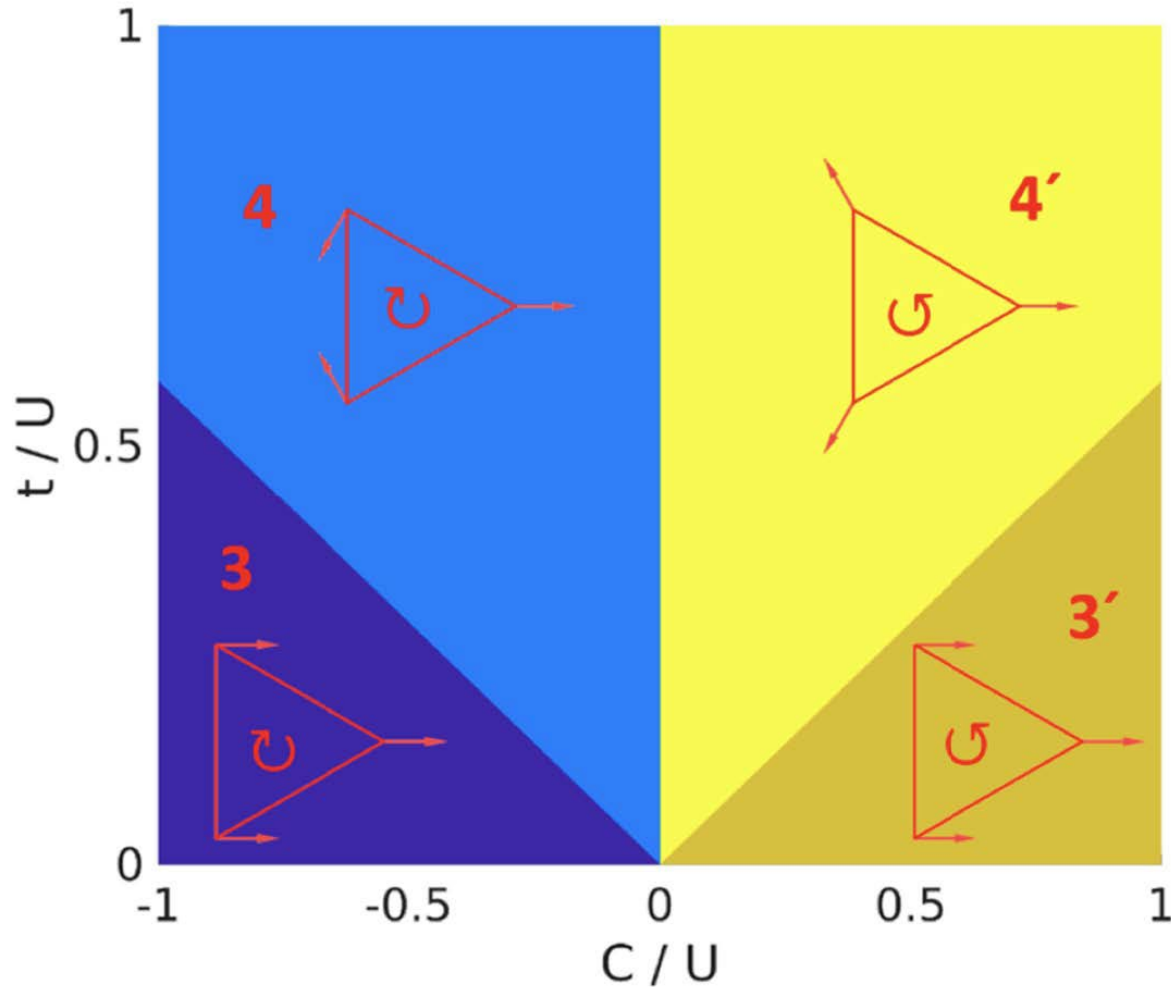
- ▶ Play a minor role for total energies
(contribute only within 2nd order perturbation theory)
- ▶ Become more important after crossover into
strongly correlated regime (symmetry!)
- ▶ The approximations considered appear reasonable
as long as the correlations remain moderate

E. A. Pluhar and C. A. Ullrich, *Phys. Rev. B* **100**, 125135 (2019)



- ▶ **Consider intrinsically noncollinear systems**
- ▶ **Role of correlations and xc torques?**
- ▶ **Symmetries and symmetry breaking: extra sensitivity?**

D. Hill, J. Shotton, and C. A. Ullrich, Phys. Rev. B **107**, 115134 (2023)



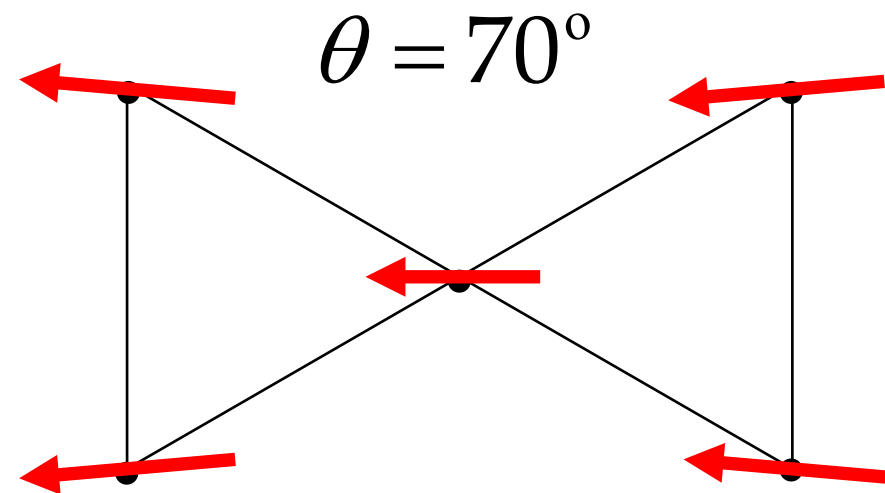
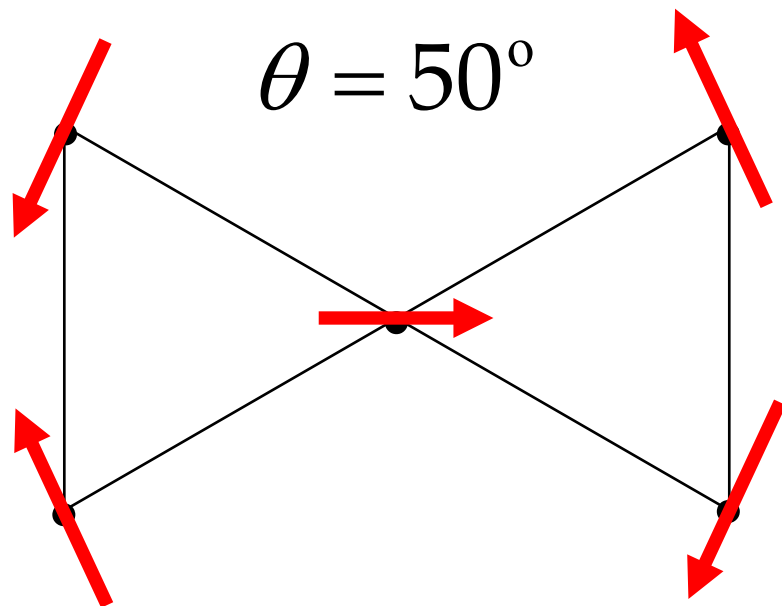
Hopping term with spin-orbit coupling strength C :

$$-t_{SOC} \sum_{\langle k,l \rangle \sigma} e^{-i\sigma\theta} \hat{c}_{k\sigma}^+ \hat{c}_{l\sigma} + h.c.$$

$$t_{SOC} = |t + iC|$$

$$t = t_{SOC} \cos(\theta)$$

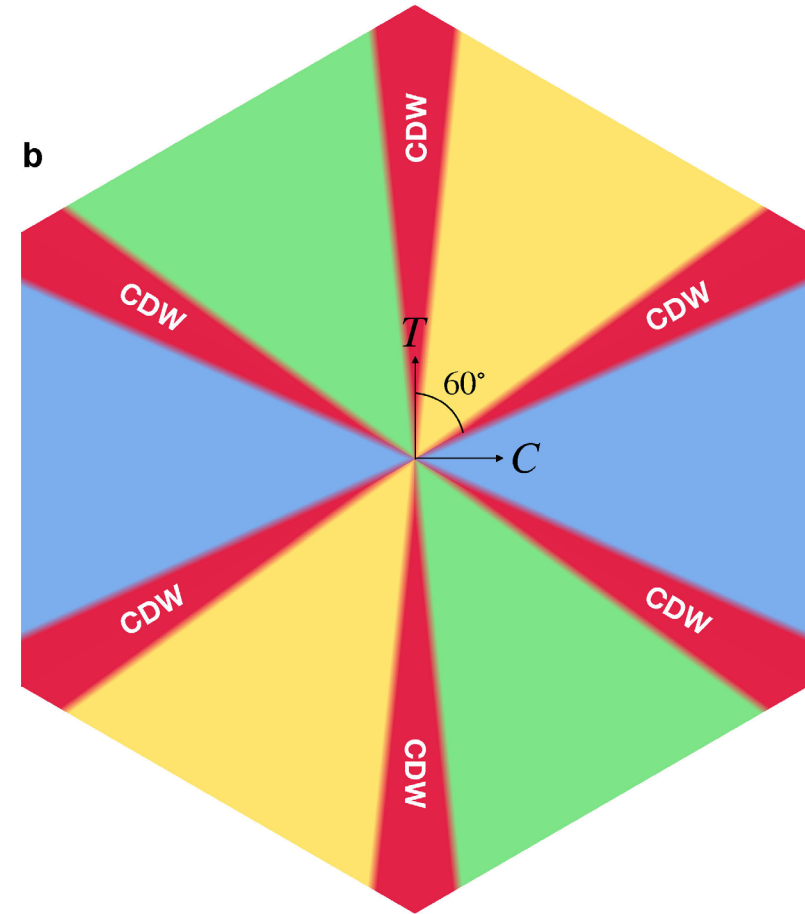
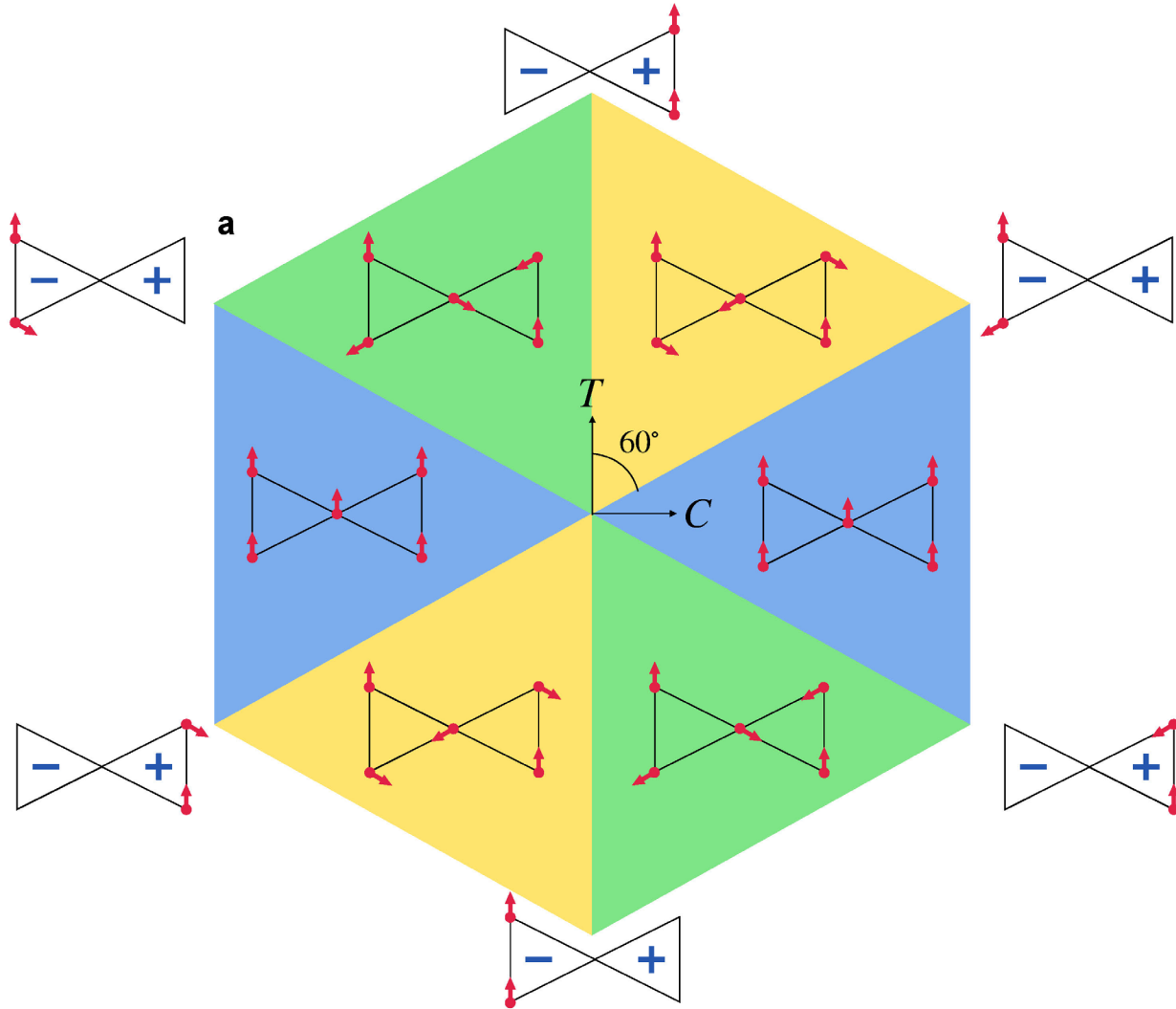
$$C = t_{SOC} \sin(\theta)$$

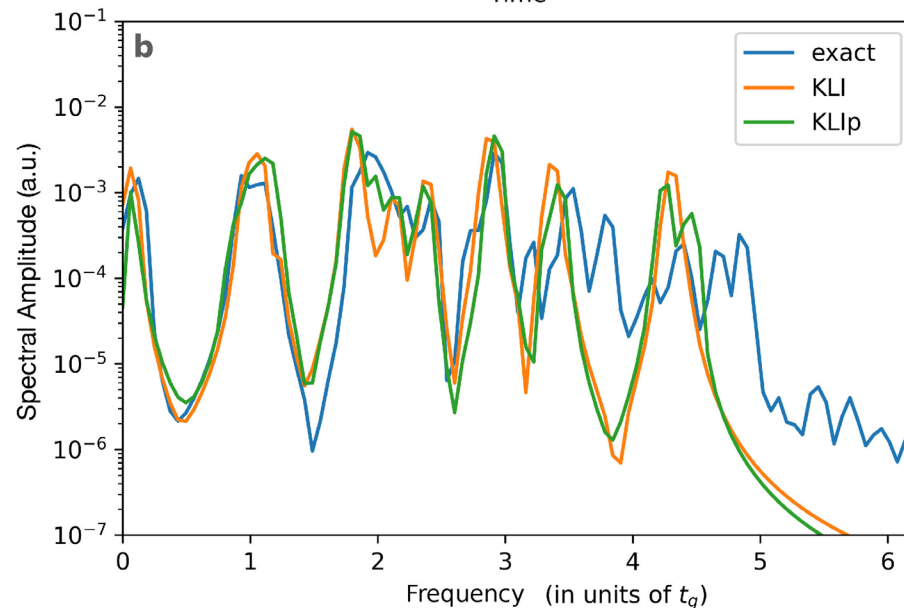
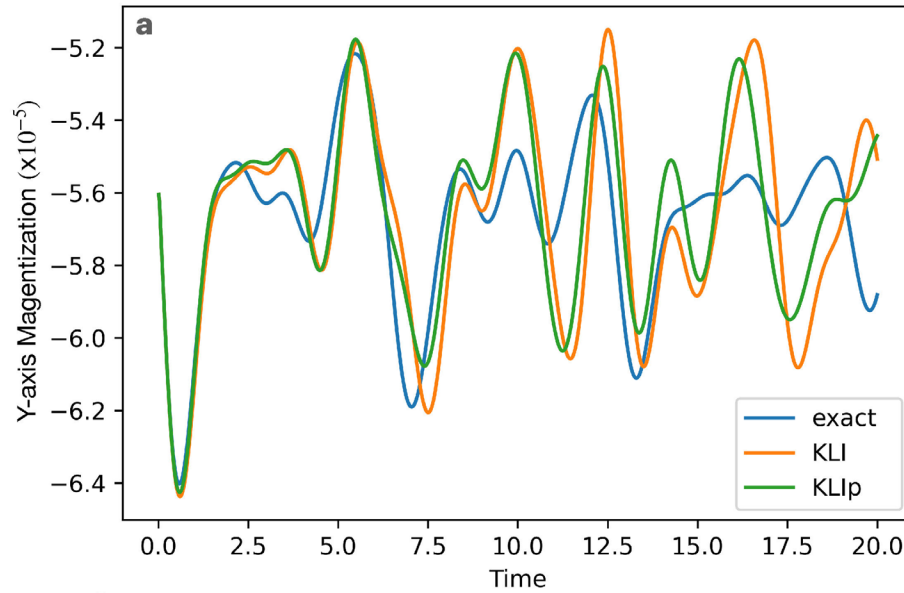


► Phase diagram boundaries at $\theta_{crit} = \frac{n\pi}{3}$ (smooth transition)

► EXX phase diagram: basic features OK, but breaks symmetry prematurely \rightarrow phase transition region too broad.

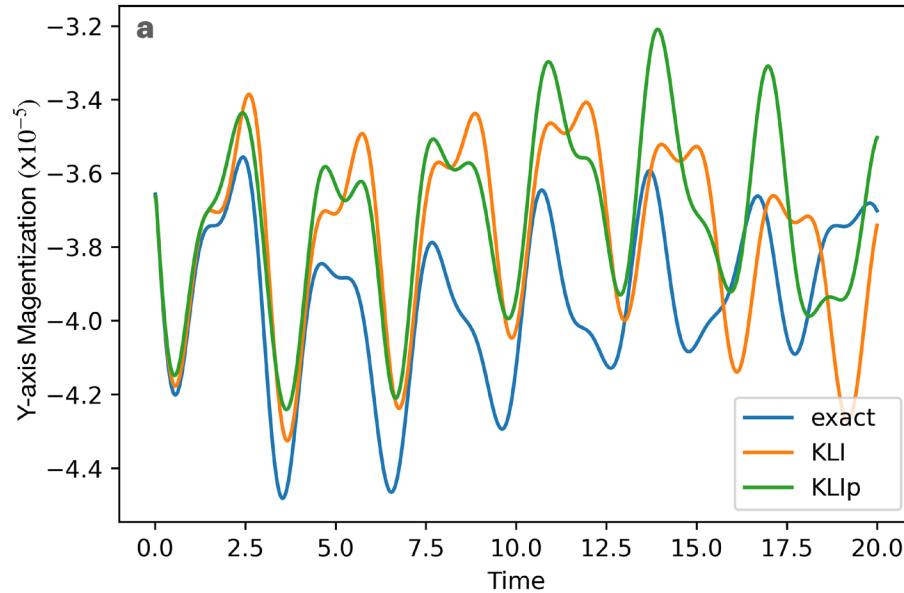
Strongly correlated around phase boundaries!





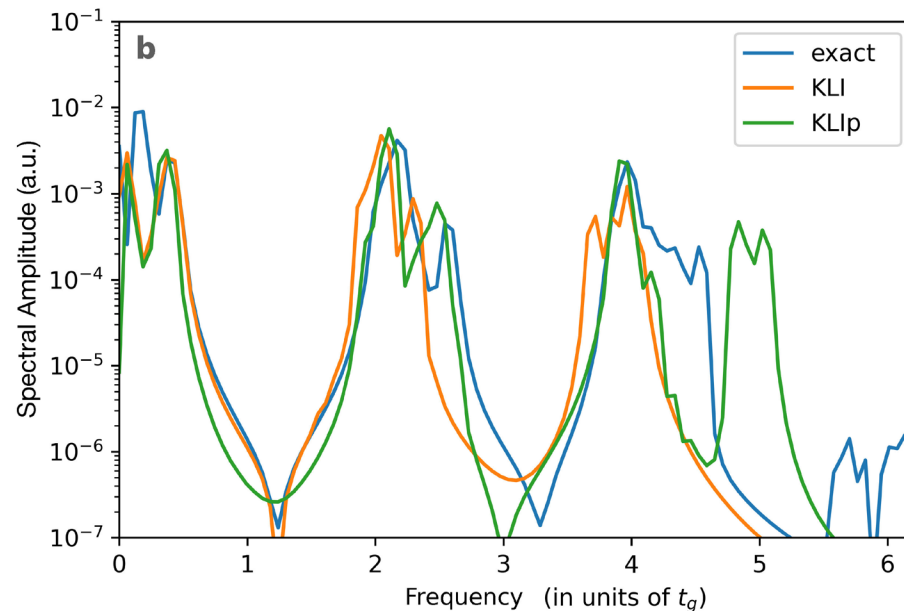
$$\theta = 30^\circ \quad U_0 = 1$$

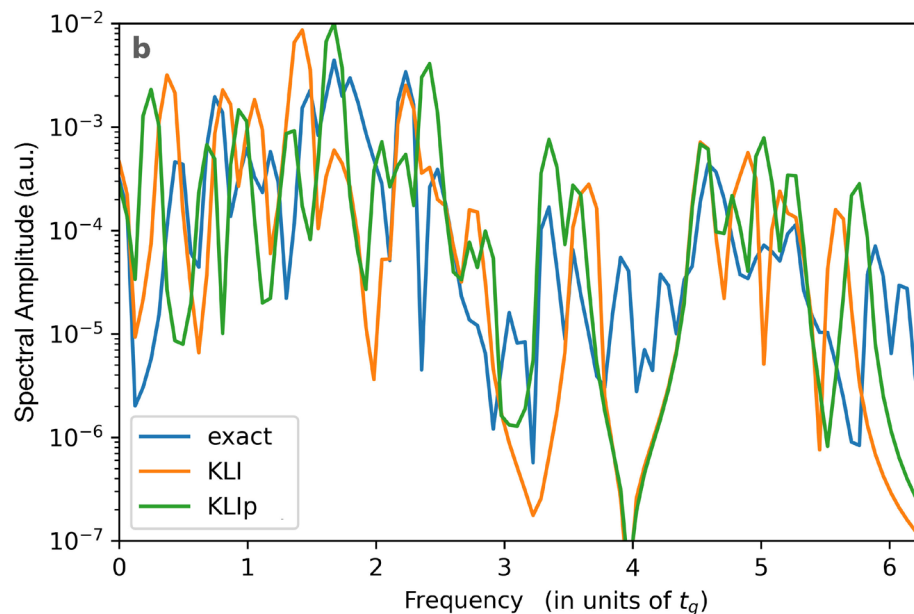
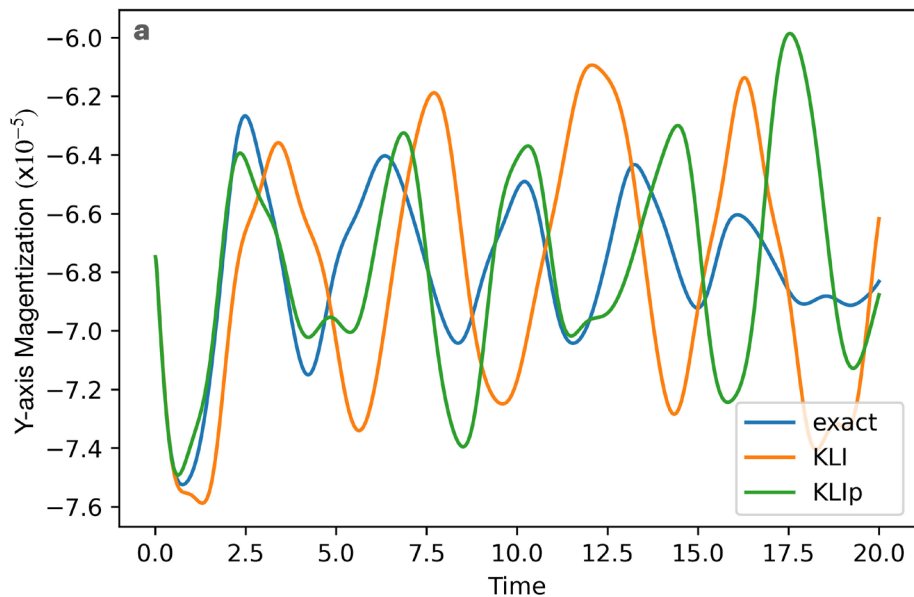
- Construct initial KS system to reproduce exact (n, m)
 - Propagate with X-only KLI and KLI_projected (no torque)
 - Free propagation after short-pulse excitation
- Weak correlation
 - good agreement with exact solution
 - torques not very important



$$\theta = 60^\circ \quad U_0 = 1$$

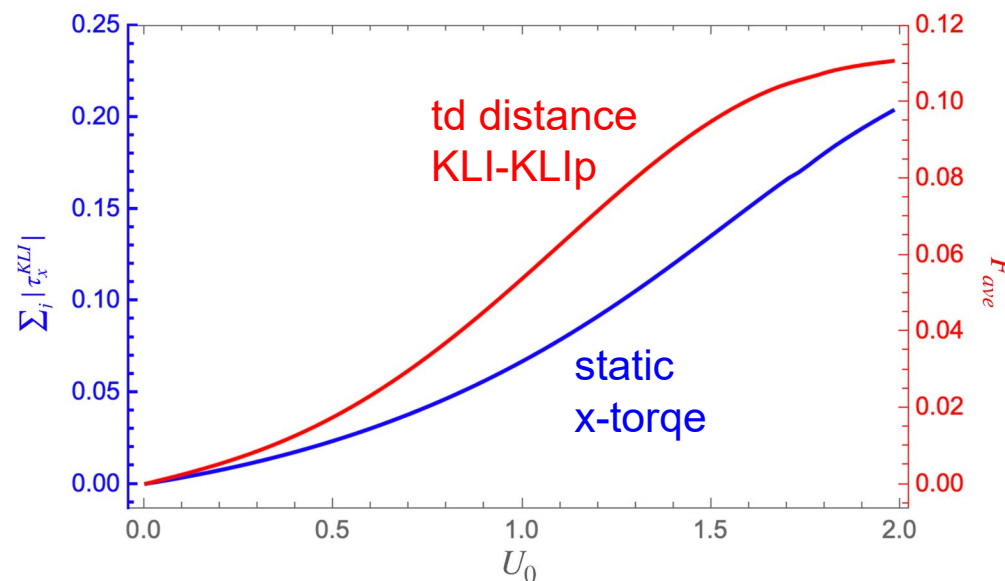
- In phase-boundary region: strongly correlated
- exchange torques relatively weak
- System dominated by symmetry-broken CDW state





$$\theta = 30^\circ \quad U_0 = 3$$

- Moderate correlation
- Projected solution (no torques) agrees better!
- X and C torques counteract. (recall Cr_3 example)



(TD)DFT for noncollinear magnetism

- ▶ Formal framework well understood
- ▶ New XC functionals:
 - *orbital based (unbiased, flexible)*
 - *MGGA (numerically well behaved, cheaper)*
- ▶ Role of XC torques:
 - *can be important for strong interactions*
 - *more tests needed (dynamics!)*

Many potential applications

- ▶ Unconventional spin structures (e.g. skyrmions)
- ▶ Ultrafast dynamics, magnonics

Challenges

- ▶ Magnetic materials can be complex (YIG, ferrites, perovskites, heterostructures)
- ▶ May need large unit cells or supercells
- ▶ Correlations can be significant

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